ACCP and atomic properties of composites and monoid domains

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Abstract

In this paper I consider ACCP and atomic properties for composites and monoid domains with different configurations of sets (domains, fields). The paper provides possible characterization of composites and monoid domains with respect to ACCP and atomic properties. Also I consider examples of primary ideals in composites.

1 Introduction

By the ring we mean a commutative ring with unity. Let R be a ring. We denote by R^* the group of all invertible elements of R. The set of all irreducible elements of R will be denoted by $\operatorname{Irr} R$.

The main motivation of this paper is description of some algebraic objects in the language of commutative algebra. This paper begins as a second (first [19]) a series of results closely related to commutative algebra.

D.D. Anderson, D.F. Anderson, M. Zafrullah in [12] called object $A+XB[X]$ as a composite for $A \subset B$ fields. If B be a domain and M be an additive cancellative monoid we can define a monoid domain $B[M] = \{a_0X^{m_0} + \cdots + a_nX^{m_n} : a_0, \ldots, a_n \in B, m_1, \ldots m_n \in M\}$. Monoid domains appear in many works such that [15], [16].

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There are many works where composites are used as examples to show some properties. But the most important works are presented below.

In 1976 [3] authors considered the structures in the form $D + M$, where D be a domain and M be a maximal ideal of ring R with $D \subset R$. Later (1.3), we could prove that in composite in the form $D + XK[X]$, where D be a domain, K be a field with $D \subset K$, that $XK[X]$ be a maximal ideal of $K[X]$. Next, Costa, Mott and Zafrullah ([4], 1978) considered composites in the form $D + XD_S[X]$, where D be a domain and D_S be a localization of D relative to the multiplicative subset S. In 1988 [8] Anderson and Ryckaert studied classes groups $D + M$. Zafrullah in [9] continued research on structure $D + XD_S[X]$ but he showed that if D be a GCD-domain, then the behaviour of $D^{(S)} = \{a_0 + \sum a_i X^i \mid a_0 \in D, a_i \in D_S\} = D + XD_S[X]$ depends upon the relationship between S and the prime ideals P of D such that D_P be a valuation domain (Theorem 1, [9]). Fontana and Kabbaj in 1990 ([11]) studied the Krull and valuative dimensions of composite $D + XD_S[X]$. In 1991 there was an article ([12]) that collected all previous results about composites and authors began to create a further theory about composites creating results. In this paper, the considered structures were officially called composites. After this article, various minor results appeared. But the most important thing is that composites have been used in many theories as examples. That is why I decided to examine all possible properties of composites for commutative algebra. The first results are in [19] and the next ones are in this paper.

In [19] I examined many properties. I will list the most important of them that may be related to this article.

Proposition 1.1. *Let* $f = a_0 + a_1X + \dots a_{n-1}X^{n-1} + a_nX^n + \cdots + a_mX^m \in T_n$ *(*$T_n = A_0 + A_1X + \cdots + A_{n_1}X^{n-1} + X^nB[X]$*,* $A_0, A_1, \dots, A_{n-1}$ *are subdomains of a field* B*), where* $0 \leq n \leq m$ *and* $a_i \in A_i$ *for* $i = 0, 1, \dots, n$ *and* $a_j \in B$ *for* $j = n, n+1, \dots, m$.

- *(i)* $f \in T_n^*$ *if and only if* $a_0 \in A_0^*$ *and* $a_1, a_2, \dots, a_m$ *are nilpotents.*
- *(ii)* f *is a nilpotent if and only if* $a_0, a_1, \dots, a_m$ *are nilpotents.*

Proof. [19] Proposition 2.6. □

Proposition 1.2. *Let* B *be a domain and* $f = a_{m_1}X^{m_1} + a_{m_2}X^{m_2} + \cdots + a_{m_n}X^{m_n} \in B[M]$*, where* $m_1, m_2, \dots, m_n \in M$ *and* $a_{m_1}, a_{m_2}, \dots, a_{m_n} \in B$*.*

- *(i)* $f \in B[M]^*$ *if and only if there exist* $m_i \in M$ *and* $a_{m_i} \in B^*$ *such that* $m_i = 0$ *and for every* $k \neq m$ *we have* a_k *be nilpotents.*

(ii) *f be a nilpotent if and only if $a_{m_1}, a_{m_2}, \ldots, a_{m_n}$ are nilpotents.*

Proof. [19] Proposition 2.8. □

Theorem 1.3. *Consider $T = A + XB[X]$, where A be a subfield of B; $T_n = A_0 + A_1X + A_2X^2 \cdots + A_{n-1}X^{n-1} + X^nB[X]$, where $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$ be fields. Then*

(i) *every nonzero prime ideal of T (T_n, respectively) is maximal;*

(ii) *every prime ideal P different from $XB[X]$ (in T) is principal;*

(iii) *every prime ideal P different from $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ (in T_n) is principal;*

(iv) *T is atomic, i. e., every nonzero nonunit of T is a finite product of irreducible elements (atoms);*

(v) *T_n is atomic.*

Proof. [19] Theorem 2.10. □

Since we are considering the properties of ACCP and atomicity, it is worth looking at the properties of GCD (greatest common divisor) and pre-Schreier.

Recall any unique factorization domain (UFD) be a GCD-domain, and any GCD-domain be a pre-Schreier domain. But if assume atomic and pre-Schreier, then we have UFD.

Example 1.4. T, T_n (See Theorem 1.3) are no GCD-domains. Let $f = a_1 + b_1X, g = a_2 + b_2X$, where $a_1, a_2 \in A, b_1, b_2 \in B$ with $A + XB[X]$. Then $\gcd(f, g) = \dfrac{a_1b_2 - a_2b_1}{b_2}$. We see that $\gcd(f, g) \in B \setminus A$.

More information about GCD-domains we can see in, e.g. [6], [13], [14].

Recall that a domain R be a pre-Schreier domain if every element $a \in R$ is a primal, i.e. for every elements $b, c \in H$ if $a \mid bc$ then there exist $a_1, a_2 \in R$ such that $a_1 \mid b, a_2 \mid c, a = a_1a_2$.

More information about Schreier and pre-Schreier domains we can see in many works, e.g. in [5], [13], [14], [7], respectively.

Lemma 1.5. *If $A \subset B$ be fields, then $A + XB[X]$ be a pre-Schreier domain. If $A_0 \subset A_1 \subset \ldots A_{n-1} \subset B$ be fields, then $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ be also a pre-Schreier domain.*

Proof. [19] Lemma 2.13. □

In the first chapter I present ACCP condition in composites and monoid domains. Recall, we say that a ring R satysfying ACCP condition (in short: has ACCP) if each increasing sequence of principal ideals is stationary. In Theorem 2.4 and Theorem 2.11 we have characterizations of composites and monoid domains satysfying ACCP condition. Theorem 2.11 is answer to Kim's hypothesis in [15] Question 1.6.

In the second chapter in a monoid domains I present atomic condition in a characterization in Theorem 3.2. Recall, we say that a ring R be atomic if every nonzero nonunit element of R can be written as a finite product of irreducible elements (also called atoms). If R has ACCP then R be atomic. The example of atomic domain which is not ACCP we can see in [17].

The third chapter is devoted to primary ideals.

The structures described in this paper often appear in the form of examples in many works. The aim of this work will be to examine as much as possible properties and applications in commutative algebra.

2 ACCP properties

In [12] Example 5.1 showed an example of an integral domain R which satisfies ACCP, but whose integral closure does not satisfy ACCP. It mean $R = \mathbb{Z} + X\overline{\mathbb{Z}}[X]$, where $\overline{\mathbb{Z}}$ be the ring of all algebraic integers. An R satisfies ACCP. For if not, then there is an infinite properly ascending chain of pricipal ideals of R. Since the degrees of the polynomials generating these principal ideals are nonincreasing, the degrees eventually stabilize. The principal ideals in $\overline{\mathbb{Z}}$ are generated by the leading coefficients of these polynomials gives an infinite ascending chain $a_1\overline{\mathbb{Z}} \subsetneq a_2\overline{\mathbb{Z}} \subsetneq ...$ where each $a_n/a_{n+1} \in \mathbb{Z}$. Thus all $a_n \in \mathbb{Q}[a_1]$. Let $A = \overline{\mathbb{Z}} \cap \mathbb{Q}[a_1]$. Then $a_1A \subsetneq a_2A \subsetneq \cdots \subsetneq A$, a contradiction since A is Dedekind.

Note that for R a ring between $A[X]$ and $B[X]$ (A be an integral domain, B be a field such that $A \subset B$), R has ACCP if and only if for every $n \geqslant 0$, any ascending chain of principal ideals generated by polynomials of degree n terminates. If B be a qoutient field of A, the Proposition 5.2 [12] may be used to show that ring R satisfies ACCP. But we can minimalize assumptions by composites.

Proposition 2.1. *Let A be an integral domain, B be a field such that $A \subset B$. Let R be a ring with $A[X] \subseteq R \subseteq B[X]$. Then R has ACCP if and only if $R \cap B$ has ACCP and for each ascending chain of polynomials $f_1R \subseteq f_2R \subseteq f_3R \subseteq \dots$ where $f_i \in R$ all have the same degree, then there is $d \in (R \cap B) \setminus \{0\}$ such that $df_i \in (R \cap B)[X]$.*

Proof. (⇒) Since $(R \cap B)^* = R^* \cap B$, R has ACCP implies $R \cap B$ has ACCP. The chain $f_1R \subseteq f_2R \subseteq \dots$ be stationary, say $f_nR = f_{n+1}R = \dots$. So $f_{n+1} = u_i f_i$, where u_i is a unit of $R \cap B$. Since $f_n \in B[X]$, there exists a $0 \neq a \in A \subseteq R \cap B$ with $af_n \in A[X] \subseteq R$. But then coefficients of $af_{n+1} = u_i df_n$ all lie in $R \cap B$.

(⇐) Let $f_1R \subseteq f_2R \subseteq \dots$ be an ascending chain in R. Since $\deg f_{i+1} \leqslant \deg f_i$, eventually f_i have the same degree, so without loss of generality, we can assume that $\deg f_1 = \deg f_2 = \dots$. By hypothesis there exists $0 \neq a \in R \cap B$ with $af_i \in (R \cap B)[X]$. Now $f_iR \subseteq f_{i+1}R$ implies $f_i = f_{i+1}b$, where $b \in R$ has degree 0, so $b \in R \cap B$. Hence $af_i(R \cap B)[X] \subseteq af_{i+1}(R \cap B)[X]$. But $R \cap B$ has ACCP and hence so does $(R \cap B)[X]$. So for large n, $f_n(R \cap B)[X] = f_{n+1}(R \cap B)[X] = \dots$, and hence $f_nR = f_{n+1}R = \dots$. □

Proposition 2.2 shows that between $A[X]$ and $A + XB[X]$ we can find a structure which satysfying ACCP condition.

Proposition 2.2. *Let A be an integral domain, B be a field such that $A \subset B$. Let C be a domain such that $A[X] \subseteq C \subseteq A + XB[X]$. Suppose that for each $n \geq 0$, there exists $a_n \in A \setminus \{0\}$ for all $f \in C$ with $\deg f \leq n$. Then C has ACCP if and only if A has ACCP.*

Proof. This is a special case of Proposition 2.1. The second part of this proof: let us call R a bounded factorization domain (BFD) if for each nonzero nonunit $a \in R$, there exists a positive integer $N(a)$, so that if $a = a_1 \dots a_s$ where each a_i is nonunit, then $s \leqslant N(a)$. It is very known fact that a BFD has ACCP but the converse is false. In the proof suppose that A be a BFD. Let $0 \neq f \in C$ have a degree n and leading coefficient b. Write $f = g_1 \dots g_s g_{s+1} \dots g_m$, where $g_1, \dots, g_m \in C$ are nonunits with $g_1, \dots, g_s \in A$ and $g_{s+1}, \dots, g_m \in C$ have a degree $\geqslant 1$. Now $g_{s+1} \dots g_m$ has a degree n, so $m - s \leqslant n$. Also, $r_n g_{s+1} \dots g_m \in A[X]$, say it has leading coefficient $c \in A$. Then $r_n b = g_q \dots g_s c$. But A is a BFD, so there is a bound on the number of factors for $r_n b$ and hence on s. Thus the m if $f = g_1 \dots g_s g_{s+1} \dots g_m$ has an upper bound. Conversely, if C be a BFD, easy to see that A be a BFD without any additional hypothesis on C. □

The above Proposition is not obvious for arbitrary composition. This would be a valuable remark, as it would allow we to choose the smallest possible composite.

Question 1: For subdomains $A_0, A_1, \dots, A_{n-1}$ of a field B, is the Proposition 2.2 valid for such domain C satysfying $A_0[X] \subseteq C \subseteq A_0 + A_1X + \dots + A_{n-1}X^{n-1} + X^nB[X]$, where the condition $A_0 \subset A_1 \subset \dots \subset A_{n-1} \subset B$ holds or not?

Kim in [15] proved very serious fact about ACCP monoid domain.

Lemma 2.3. *Let A be a domain. Then A has ACCP if and only if $A[X]$ has ACCP.*

Proof. [15], Corollary 2.2. Can be easily proved by comparing degrees. □

It also turn out that ACCP property moves between A and $A + XB[X]$. This is important because we do not have to choose a general polynomial, and we can limit the inclusion to the smallest composite needed. Such a significant limitation of a polynomial to a composite is important, e.g. in Galois theory in commutative rings.

Theorem 2.4. *Let A be an integral domain, B be a field such that $A \subset B$. An A has ACCP if and only if $A + XB[X]$ has ACCP.*

Proof. From Proposition 2.2 we have $A[X] \subseteq A + XB[X] \subseteq A + XK[X]$, where K be a qoutient field of B. We can to prove that for each $n \geq 0$, there exists $a_n \in A \setminus \{0\}$ for all $f \in A + XB[X]$ with $\deg f \leq n$. Because A has ACCP then from Proposition 2.2 we get $A + XB[X]$ has ACCP. Conversly, because $A + XB[X]$ has ACCP then A has ACCP. □

The next facts are the conclusions of Theorem 2.4.

Corollary 2.5. *Let $A_0, A_1, \ldots, A_{n-1}$ be subdomains of a field B such that $A_0 \subset A_1 \subset \cdots \subset B$. Let C be a domain with $A_0[X] \subseteq C \subseteq A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. Suppose that for each $n \geq 0$, there exists $a_n \in A_0 \setminus \{0\}$ for all $f \in C$ with $\deg f \leq n$. Then C has ACCP if and only if A_0 has ACCP.*

Corollary 2.6. *Let $A_0, A_1, \ldots, A_{n-1}$ be subdomains of a field B such that $A_0 \subset A_1 \subset \cdots \subset B$. An A_0 has ACCP if and only if $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ has ACCP.*

If assume $A_0, A_1, \ldots, A_{n-1}$ are integral domains, B be a field such that $A_0, A_1, \ldots, A_{n-1} \subset B$, then the above Corollaries do not aply because such composite is not a ring ([19] Corollary 2.3).

Next Proposition is about A-valued B-polynomials. Recall that A-valued B-polynomials be a structure in the form $I(B, A) = \{f \in B[X] : f(A) \subseteq A\}$. Of course $I(B, A) \subset A + XB[X]$, where A, B are domain such that $A \subset B$.

Proposition 2.7. *Let A be an integral domain, B be a field, where $A \subset B$. For each $n \geqslant 0$, there exists $a_n \in A \setminus \{0\}$ such that $a_nf(X) \in A[X]$ for all $f(X) \in I(B, A)$ with $\deg f(X) \leqslant n$.*

Proof. For $n = 0$ we may take $a_0 = 1$. Assume that $a_{n-1} \in A \setminus \{0\}$ has been chosen such that $a_{n-1}g(X) \in A[X]$ for all $g(X) \in I(B, A)$ with $\deg g(X) \leqslant n-1$. If $A = B$, we may take $a_n = 1$. So suppose that a b_0 be a nonzero nonunit of A. Let $f(X) = c_0+c_1X+\ldots c_nX^n \in I(B, A)$ have a degree n. Now $c_0 + b_0c_1X + \ldots b_0^n c_nX^n = f(b_0X) \in I(B, A)$ as is $b_0^n f(X) = b_0^n c_0 + b_0^n c_1X + \ldots b_0^n c_nX^n$. Hence $(b_0^n - 1)c_0 + (b_0^n - b_0)c_1X + \cdots + (b_0^n - b_0^{n-1})c_{n-1}X^{n-1} = b_0^n f(X)-f(b_0X) = g(X) \in I(B, A)$. By induction, $a_{n-1}g(X) \in A[X]$, that is, $a_{n-1}(b_0^n - b_0^i)c_i \in A$ for $i = 0, 1, \ldots, n-1$. Put $a_n = a_{n-1}\prod_{i=0}^{n-1}(b_0^n - b_0^i) \in A$. Since b_0 be a nonzero nonunit, each $b_0^n - b_0^i \neq 0$, so $a_n \neq 0$. Certainly $a_nc_i \in R$ for $i = 0, 1, \ldots, n-1$. Now $c_0 + c_1 + \ldots c_n = f(1) \in A$, so $a_nc_0 + \cdots + a_nc_n = a_nf(1) \in A$, hence $a_nc_n \in A$. So $a_nf(X) \in A[X]$. □

Corollary 2.8. *Let A be an integral domain, B be a field such that $A \subset B$. Then $I(B, A)$ satisfies ACCP if and only if A satisfies ACCP.*

Proof. Combine Proposition 2.2 and Proposition 2.7. □

Next lemmas coming from Kim [15] are results about ACCP properties in monoid domains.

Lemma 2.9. *Let $S \subseteq T$ be an extension of torsion-free cancellative monoids. If T satisfies ACCP and $T^* \cap S = S^*$, then S satisfies the ACCP.*

Proof. [15] Proposition 1.2. (1). □

Lemma 2.10. *Let D be an integral domain, S a torsion-free cancellative additive monoid, and $D[S]$ the monoid domain. If $D[S]$ satisfies ACCP, then D and S satisfy ACCP.*

Proof. [15], Proposition 1.5. □

Next Theorem is the answer about question from Kim [15] Question 1.6. In [15] Proposition 1.5 (1) we have an implication. Kim asked that are the sufficient conditions in [15] Proposition 1.5 (1) for the monoid domain to satisfy ACCP, necessary.

Theorem 2.11. *Let A be an integral domain and B be a field such that $A \subset B$ and $A[S]^* = B[S]^*$. Let S be a torsion-free cancellative monoid. Both A and $B[S]$ satisfy ACCP if and only if $A[S]$ satisfies ACCP.*

Proof. ($\Rightarrow$) The proof is similar to [15], Proposition 1.5.

($\Leftarrow$) From Lemma 2.10, since $A[S]$ has ACCP, then A has ACCP. Now, consider $f_1, f_2, \cdots \in B[S]$ such that $\ldots, f_3 \mid f_2, f_2 \mid f_1$. Without loss of generality, we can assume that $f_1, f_2, \dots \in \operatorname{Irr} B[S]$ because every ACCP-domain is atomic. Since $A^* = B^*$, so $f_1, f_2, \dots \in \operatorname{Irr} A[S]$. By assumption $A[S]$ has ACCP, so there exists $n \geqslant 1$ such that $f_n \mid f_{n-1}, \dots, f_3 \mid f_2, f_2 \mid f_1$. We get $(f_1) \subseteq (f_2) \subseteq \dots \subseteq (f_n) = (f_{n+1}) = \dots$ in $B[S]$ which is stationary. □

In the below we have known fact from [17].

Lemma 2.12. *Let D be an integral domain. A D satisfies ACCP if and only if $D[X]$ satisfies ACCP.*

3 Atomic properties

In this section we have results about atomicity in a monoid domain.

Lemma 3.1. *Let D be an integral domain, S a torsion-free cancellative monoid, and $D[S]$ the monoid domain. If $D[S]$ be atomic, then D and S be atomic.*

Proof. [15], Proposition 1.4. □

Next Theorem is similarly to 2.11.

Theorem 3.2. *Let A be an integral domain and B be a field such that $A \subset B$ with $A[S]^* = B[S]^*$. Let S be a torsion-free cancellative monoid. Both A and $B[S]$ be atomic if and only if $A[S]$ be atomic.*

Proof. ($\Rightarrow$) Since $B[S]$ be atomic, then consider $f = g_1 g_2 \dots g_n \in B[S]$, where $g_1, g_2, \dots, g_n \in \operatorname{Irr} B[S]$. Hence from assumption we have $g_1, g_2, \dots, g_n \in \operatorname{Irr} A[S]$. Then $A[S]$ is atomic.

($\Leftarrow$) From Lemma 3.1 since $A[S]$ be atomic, then A and S be atomic. Now consider $f = g_1 g_2 \dots g_n \in A[S]$, where $g_1, g_2, \dots, g_n \in \operatorname{Irr} A[S]$, because $A[S]$ be atomic. Then $g_1, g_2, \dots, g_n \in \operatorname{Irr} B[S]$, hence $B[S]$ be atomic. □

Anderson, Anderson and Zafrullah asked in [10] (Question 1) is $R[X]$ atomic when R is atomic. I say no. I have no example but we can deduce from well known facts:

Suppose that $R[X]$ is not atomic. We want to get R is not atomic. Since $R[X]$ is not atomic then $R[X]$ has no ACCP. Hence R has no ACCP which it does not imply R is not atomic because there exists an example atomic domain which is not ACCP.

Converse, if R is not atomic, then R has no ACCP. Hence $R[X]$ has no ACCP which it does not imply $R[X]$ is atomic.

4 Primary ideals

The issue of primary ideals is a very important part, e.g. in the consideration of noetherian rings, which is why I decided to study composites in this respect. Recall that an ideal I of the ring R is called primary if I satisfies the following implication for any $a, b \in R$:

$$ab \in I \Rightarrow a \in I \vee b^n \in I.$$

Let's start from easy Lemma. Next show primary ideal on composites.

Lemma 4.1. *Let $A \subset B$ be integral domains. If A be an ideal of B. Then $A[X]$ be an ideal of $A + XB[X]$.*

Proposition 4.2. *Let A and B be integral domains. If A be an ideal of B. Then $A[X]$ be an primary ideal of $A + XB[X]$.*

Proof. From lemma 4.1 we have $A[X]$ is an ideal of $A+XB[X]$. The condition "$A[X]$ is a primary ideal of $A + XB[X]$" is equivalent to "any nonzero zero divisor of $A + XB[X]/A[X]$ is a nilpotent. Note that $A + XB[X]/A[X] \cong X(B\setminus A)[X]$. Let's consider any nonzero zero divisor $f \in X(B\setminus A)[X]$. Then there exist a $b \in B \setminus A, b \neq 0$ such that $bf = 0$. Raising to some power n we have $b^n f^n = 0$. If $b^n = 0$ then $b = 0$, but $b \neq 0$. A contradiction. Hence $f^n = 0$. Therefore in $A + XB[X]/A[X]$ all zero divisors are nilpotents, i.e. $A[X]$ is a primary ideal of $A + XB[X]$. □

Lemma 4.3. *Let $A_0, A_1, \dots, A_{n-1}$ be subdomains of a field B with $A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset B$. If A_0 be an ideal of A_1, A_1 be an ideal of A_2, ..., A_{n-2} be an ideal of A_{n-1}, A_{n-1} be an ideal of B, then $A_0[X]$ be an ideal of $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$.*

Proposition 4.4. *Let $A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset B$ be domains. If A_0 be an ideal of A_1, A_1 be an ideal of A_2, ..., A_{n-2} be an ideal of A_{n-1}, A_{n-1} be an ideal of B, then $A_0[X]$ be an primary ideal of $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$.*

Proof. From lemma 4.3 we have $A_0[X]$ is an ideal of $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. Proceed similarly to 4.2. □

Question: How to find primary ideals in any monoid domain?

5 Examples

Example 5.1. At the beginning of the second chapter I gave an example an integral domain R which satisfies ACCP, but whose integral closure does not satisfy ACCP. It means $R = \mathbb{Z} + X\overline{\mathbb{Z}}$.

Example 5.2. Recall that a domain R is called a half-factorial domain (HFD) if R is atomic and for each nonzero nonunit $x \in R$, $x = x_1 \dots x_m = y_1 \dots y_n$ where x_i, y_j are all irreducible for $i = 1, \dots, m, j = 1, \dots, n$, implies that $m = n$. A HFD domain satisfies ACCP.

Example 5.3. Let $R = \mathbb{R} + X\mathbb{C}[X]$. So R is a HFD, so has ACCP, then atomic.

Example 5.4. ([2]) Let F be a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/3, 1/(2 \cdot 5), \dots, 1/(2^j p_j), \dots\}$, where $p_0 = 3, p_1 = 5, \dots$ is the sequence of odd primes. Let R be the monoid domain $F[X;T] = F[T]$ and $N = \{f \in R \mid f \text{has nonzero constant term}\}$. Then $F[T]_N$ is an atomic domain which does not satisfy ACCP.

Example 5.5. Let K be a field and T the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/2, 1/3, 1/5, \dots, 1/p_j, \dots\}$, where p_j is the jth prime. Then the monoid domain $R = K[T]$ satisfies ACCP.

For a $0 \neq f = b_1X^{a_1} + \dots b_nX^{a_n} \in R$ with $a_1 < \dots < a_n$ and $b_n \neq 0$, write $\beta(f) = a_n$. If ACCP fails, the there is a strictly increasing chain $(f_1) \subset (f_2) \subset \dots$ of principal ideals in R. Then each $f_n = f_{n+1}g_{n+1}$ for some nonunit $g_{n+1} \in R$. Hence each $\beta(f_n) = \beta(f_{n+1}) + \beta(g_{n+1})$, and each term is positive. Then in T, we have $\beta(f_1) > \beta(f_2) > \dots$ with each $\beta(f_n) - \beta(f_{n+1}) \in T$, but this is impossible by the above-mentioned unique representation of each nonzero $a \in T$.

Example 5.6. ([10]) Let K be a field, $T = \{q \in \mathbb{Q} \mid q \geqslant 1\} \cup \{0\}$ an additive submonoid of $\mathbb{Q}^+$, and $R = K[T]$ the monoid domain. Then $R_S = K[\mathbb{Q}]$, where $S = \{X^t \mid t \in T\}$, is not atomic since R_S is a GCD-domain, but R_S does not satisfy ACCP.

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