On square-free and radical factorizations and relationships with the Jacobian Conjecture

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Abstract

We discuss various square-free and radical factorizations and existence of some divisors in commutative cancellative monoids in the context of: atomicity, ascending chain condition for principal ideals, a pre-Schreier property, a greatest common divisor property and a greatest common divisor for sets property. We also discuss the analogues of Jacobian conditions and their relationship to square-free and radical factorizations.

1 Introduction

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Throughout this paper by a monoid we mean a commutative cancellative monoid.

Let H be a monoid. We denote by H^* the group of all invertible elements of H.

If $a, b \in H$ are relatively primes in H, i.e. do not have a common invertible divisor of H, then we write $a \operatorname{rpr} b$. Therefore, if M be a submonoid of H and elements $a, b \in M$ are relatively primes in M, then we write $a \operatorname{rpr}_M b$.

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If a, $b \in H$ satysfying the condition $a = ub$, where $u \in H^*$, then we write $a \sim b$. Therefore, if M be a submonoid of H and elements a, $b \in M$ satysfying $a = ub$, where $u \in H^*$, then we write $a \sim_M b$.

The set of all irreducible elements (atoms) of H will be denoted by $\operatorname{Irr} H$. Recall that an element $a \in H$ is called square-free if it cannot be presented in the form $a = b^2c$, where b, $c \in H$ and $b \notin H^*$. The set of all square-free elements of H we will denote by $\operatorname{Sqf} H$.

In Theorem 8.1 we present a full characterization of submonoids M of the factorial monoid H satisfying the condition

(1) $\operatorname{Sqf} M \subset \operatorname{Sqf} H$

assuming $M^* = H^*$.

The equivalence of (1) and

(2) for every $a \in H$, $b \in \operatorname{Sqf} H$, if $a^2b \in M$, then a, $b \in M$

in [4] has been extended to the equivalence of 8 conditions. Two of these conditions represent a closure with respect to the 1s and 3s factorization (See section 4), while the closure with respect to 3s was obtained at an earlier stage of the research and published in [2].

In addition, we received a full description of such submonoids (of factorial monoid) satisfying the condition (1). They are (with an accuracy to the invertible elements) free submonoids generated by any set of pairs of relatively primes non-invertible square-free elements.

It also turned out (Theorem 8.1) that the condition

(3) $\operatorname{Irr} M \subset \operatorname{Sqf} H$

is equivalent to the conjunction of (1) and

(4) for every a, $b \in M$, if $a \operatorname{rpr}_M b$, then $a \operatorname{rpr}_H b$.

We have a transparent answer to the question of when the condition (1) be equivalent to the condition (3) (See Theorem 8.1).

A very important step in the conducted research was finding a factorial condition (Theorem 9.3) implicating the condition (3):

(5) for every $a \in H$, $b \in \operatorname{Sqf} H$, if $a^2b \in M$, then a, $ab \in M$.

A natural question arose, is it a necessary condition. The answer is negative – a counterexample was found (Example 9.2). The factorial condition to (3) is interesting, five equivalent conditions were obtained (Theorem 9.3), including closure with respect to the factorization of 2s (See section 4).

Conditions (1) and (3) are related to the assumption found in the famous Jacobian conjecture.

Conjecture 1.1. Let k be a field of characteristic 0. For every polynomials $f_1, f_2, \dots, f_n \in k[x_1, \dots, x_n]$ with $n > 1$, if

$$jac(f_1, f_2, \dots, f_n) \in k \setminus \{0\},$$

then

$$k[f_1, \dots, f_n] = k[x_1, \dots, x_n].$$

Recall a generalization of the Jacobian conjecture formulated in [4].

Conjecture 1.2. Let k be a field characteristic 0. For every polynomials f_1, $f_2, \dots, f_r \in k[x_1, \dots, x_n]$ with $n > 1$ and $r \in \{2, \dots, n\}$, if

$$\gcd(jac^{f_1, f_2, \dots, f_r}_{x_{j_1}, x_{j_2}, \dots, x_{j_r}}, 1 \leqslant j_1 < \dots < j_r \leqslant n) \in k \setminus \{0\},$$

then

$$k[f_1, \dots, f_r] \text{ is algebraically closed in } k[x_1, \dots, x_n].$$

Under the assumption that $f_1, f_2, \dots, f_r$ are algebraically independent over k, the generalized Jacobian condition (assumption of Conjecture 1.2) is equivalent to any of the following ones ([4]):

(6) every irreducible of $k[f_1, \dots, f_r]$ is square-free in $k[x_1, \dots, x_n]$,

(7) every square-free of $k[f_1, \dots, f_r]$ is square-free in $k[x_1, \dots, x_n]$.

Conditions (1) and (3) are a generalization of conditions (6) and (7) (because (1), (3) are monoid-version) and therefore we call them the analogs of the Jacobian conditions.

A side effect of the presented approach was a natural question about general relationships between square-free factorizations in different classes of monoids. Of course, these factorizations for rings of polynomials are commonly known, and it is clear that their existence and uniqueness occur in domains with uniqueness of distribution, so e.g. certain properties hold in GCD-domains. However, these relationships have not been studied so far. In this paper we consider pre-Schreier monoids, GCD-monoids, GCDs-monoids, ACCP-monoids, atomic monoids.

Recall that a monoid is called GCD-monoid, if for any two elements there exists a greatest common divisor. A monoid H is called GCDs-monoid, if there exists greatest common divisor for any subset of H. A monoid H is called a pre-Schreier monoid, if any element $a \in H$ is primal, i.e. for any b, $c \in H$ such that $a \mid bc$ there exist a_1, $a_2 \in H$ such that $a = a_1 a_2$, $a_1 \mid b$ and $a_2 \mid c$. A monoid H is called atomic, if every non-invertible element $a \in H$ be a finite product of irreducibles (atoms). A monoid H is factorial, if each non-invertible element can be written as a product of irreducible elements and this representation is unique. A monoid H is called ACCP-monoid if any ascending sequence principal ideals of H stabilizes. Recall that every factorial monoid is ACCP-monoid, and ACCP-monoid is atomic ([8]). And recall that every factorial monoid is GCDs-monoid, then GCD-monoid. And GCD-monoid is pre-Schreier ([8]). Since every pre-Schreier is AP-monoid (in such monoid an irreducible element (atom) is prime), then every atomic and AP-monoid is factorial.

In section 5 we examine the dependencies between square-free factorizations, conditions of existence of certain square-free divisors, and between square-free factorizations and conditions of existence of certain square-free divisors. The conditions for the existence of certain square-free divisors result from the appropriate factorization, and the condition for the existence of a square-free divisor in a square plays an important role in reasoning about the inclusions (1) and (3).

In this context, the concept of a radical generator is very important introduced by A. Reinhart in 2012 in [6]. The element of monoid is called radical if the principal ideal is generated by this element be a radical ideal. The set of all radical generators of a monoid H will be denoted by $\operatorname{Gpr} H$. Reinhart's explores the properties of radically factorial monoids, i.e. such that each element is a product of radical generators. He does not consider various types of radical factorization, nor relationships with square-free factorization. Let us add that the property of the radical generator (although the author does not use this name) appeared in the work of G. Angermüller published in 2017 in the Grauert-Remmert normality criterion ([1], Proposition 31).

The radical generator is square-free, so radical factorizations are square-free factorizations. Therefore, in the section 5 we also study general relationships between radical factorizations, conditions of existence of certain radical divisors, as well as between factorizations and conditions of existence of some divisors (square-free or radical).

In sections 4 and 5 we present the latest results, which include the relationships between 8 factorization and 16 conditions for the existence of the divisor.

2 Auxiliary statements

In this section we present lemmas that we will need later in this paper.

Lemma 2.1. *Let H be a monoid.*

(a) *Let $a \in \operatorname{Sqf} H$ and $b \in H$. If $b \mid a$ then $b \in \operatorname{Sqf} H$.*

(b) *Let $a \in \operatorname{Gpr} H$ and $b \in H$. If $b \mid a$ then $b \in \operatorname{Gpr} H$.*

Proof. (a) Suppose $b \notin \operatorname{Sqf} H$. Then there exists $d \in H \setminus H^*$ such that $d^2 \mid b$. Hence $d^2 \mid a$. A contradiction.

(b) Let $a \in \operatorname{Gpr} H$ and $b \mid a$. Let $c \in H$ and $b \mid c^n$ for some $n \in \mathbb{N}$. By assumption we have $a = bd$, where $d \in H$. Then $a \mid c^n d^n$ and this implies $a \mid cd$, so $b \mid c$.

□

Lemma 2.2. *Let H be a monoid. If $a \in \operatorname{Sqf} H$ and $a = b_1 b_2 \dots b_n$, then $b_i \operatorname{rpr} b_j$ for $i, j \in \{1, \dots, n\}$, $i \neq j$.*

Proof. Suppose b_i and b_j have a common non-invertible divisor d for some $i, j \in \{1, \dots, n\}$. Hence $a = b_1 b_2 \dots b_n$ is not square-free because $d^2 \mid a$. □

Lemma 2.3. *Let H be a pre-Schreier monoid.*

(a) *Let $a, b, c, d \in H$. If $ab = cd$, $a \operatorname{rpr} c$ and $b \operatorname{rpr} d$, then $a \sim d$ and $b \sim c$.*

(b) *Let $a_1, a_2, \dots, a_n, b \in H$. If $a_i \operatorname{rpr} b$ for $i = 1, 2, \dots, n$, then $a_1 a_2 \dots a_n \operatorname{rpr} b$.*

(c) *Let $a, b \in H$. If $a \operatorname{rpr} b$, then $a^k \operatorname{rpr} b^l$ for any $k, l \in \mathbb{N}$.*

(d) *Let $a_1, a_2, \dots, a_n \in H$. If $a_1, a_2, \dots, a_n \in \operatorname{Sqf} H$ and $a_i \operatorname{rpr} a_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, then $a_1 a_2 \dots a_n \in \operatorname{Sqf} H$.*

(e) *Let $a_1, a_2, \dots, a_n \in \operatorname{Sqf} H$, $b \in H$. If $a_i \operatorname{rpr} a_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$ and $a_i \mid b$ for $i = 1, 2, \dots, n$, then $a_1 a_2 \dots a_n \mid b$.*

(f) *Let $a, b_1, \dots, b_n \in H$. If $a \mid b_1 \dots b_n$, then there exist $a_1 \dots a_n \in H$ such that $a = a_1 \dots a_n$ and $a_i \mid b_i$ for $i = 1, \dots, n$.*

Proof. (a) Assume that $ab = cd$, $a \operatorname{rpr} c$ and $b \operatorname{rpr} d$. If $a = 0$ and H is not a group, then $c \in H^*$, so $d = 0$ and then $b \in H^*$.

Now, let $a, d \neq 0$. Since $a \mid cd$ and $a \operatorname{rpr} c$, we have $a \mid d$. Similarly, since $d \mid ab$ and $d \operatorname{rpr} b$, we obtain $d \mid a$. Hence, $a \sim d$, and then $b \sim c$.

(b) Induction. Let $a_i \operatorname{rpr} b$ for $i = 1, \dots, n+1$. Put $a = a_1 \dots a_n$. Assume that $a \operatorname{rpr} b$. Let $c \in H$ be a common divisor of aa_{n+1} and b. Since $c \mid aa_{n+1}$, there exist $c_1, c_2 \in H$ such that $c_1 \mid a$, $c_2 \mid a_{n+1}$ and $c = c_1 c_2$. We see that $c_1, c_2 \mid b$, so $c_1, c_2 \in H^*$, and then $c \in H^*$.

(c) Let $a, b \in H$. Assume $a \operatorname{rpr} b$. Then by (b) we get $a^k \operatorname{rpr} b$ for any $k \in \mathbb{N}$. And again by (b) we have $a^k \operatorname{rpr} b^l$ for any $l \in \mathbb{N}$.

(d) Induction. Take $a_1, \dots, a_{n+1} \in \operatorname{Sqf} H$ such that a_i rpr a_j for $i \neq j$. Put $a = a_1 \dots a_n$. Assume that $a \in \operatorname{Sqf} H$. Let $aa_{n+1} = b^2 c$ for some $b, c \in H$.

Since $c \mid aa_{n+1}$, there exist $c_1, c_2 \in H$ such that $c = c_1 c_2$, $c_1 \mid a$ and $c_2 \mid a_{n+1}$, so $a = c_1 d$ and $a_{n+1} = c_2 e$, where $d, e \in H$. We obtain $de = b^2$. By (b) we have a rpr a_{n+1}, so d rpr e. And then by (c), there exist $b_1, b_2 \in H$ such that $d \sim b_1^2$, $e \sim b_2^2$ and $b = b_1 b_2$. Since $a, a_{n+1} \in \operatorname{Sqf} H$, we infer $b_1, b_2 \in H^*$, so $b \in H^*$.

(e) Induction. Assume the assertion for n. Consider $a_1, \dots, a_n, a_{n+1} \in \operatorname{Sqf} H$, $a_i \operatorname{rpr} a_j$ for $i \neq j$, and $b \in H$ such that $a_i \mid b$ for $i = 1, \dots, n+1$. Put $a = a_1 \dots a_n$. Then, by induction hypothesis, $a \mid b$, so $b = ac$ for some $c \in H$. Moreover, $a \operatorname{rpr} a_{n+1}$ by (b). Since $a_{n+1} \mid ac$ we obtain $a_{n+1} \mid c$, and then $aa_{n+1} \mid ac$.

(f) Simple induction.

□

In the following Proposition we have a very important property in a pre-Schreier monoid.

Proposition 2.4. *Let H be a pre-Schreier monoid. Then*

$$\operatorname{Gpr} H = \operatorname{Sqf} H.$$

Proof. Let $a \in \operatorname{Sqf} H$. Assume that $a \mid b^n$ for some $b \in H$ and $n \in \mathbb{N}$. Then, by Lemma 2.3 (f), there exist $a_1, \dots, a_n \in H$ such that $a = a_1 \dots a_n$ and $a_i \mid b$ for $i = 1, \dots, n$. Observe that $a_1 \dots a_n \in \operatorname{Sqf} H$ and $a_i \operatorname{rpr} a_j$ for $i \neq j$, by Lemma 2.2, so $a_1 \dots a_n \mid b$ by Lemma 2.3 (e). □

Lemma 2.5. *Let H be a GCDs-monoid and $a \in H$. Let $X \subset H$ be any non-empty subset of set of divisors of a. Then there is* $\operatorname{GCD}(X)$.

Proof. Let $Y = \{d \in H \mid \exists c \in X : a = cd\}$. Denote by e a greatest common divisor of the set Y. Then e divides every element of the set Y, so by definition of Y we get $e \mid a$. We have $a = ef$, where $f \in H$. We will show $f = \mathrm{GCD}(X)$.

First we prove that f is least common multiple of elements of the set X. Consider any element $c \in X$. Since $c \mid a$, then $a = cd$, where $d \in H$. We have $d \in Y$, so $d = eg$, where $g \in H$. Thus, since $d = eg$, then $cd = ceg$, and since $ef = a = cd$, then $ef = ceg$. Then $f = cg$, so $c \mid f$.

Now, we will show that every least common multiple of elements of X is the multiple of element f. Consider any element $c \in X$ such that $a = cd, d \in Y$. We know that $c \mid h$, so $cd \mid hd$, hence $a \mid hd$. Let $Z = \{bh, b \in Y\}$. Then we have $\mathrm{GCD}(Z) = h \cdot \mathrm{GCD}(Y) = he$. Since $a \mid hl$, then $a \mid eh$. We know $a = ef$, hence $ef \mid eh$, so $f \mid h$. □

Lemma 2.6. *Let H be a monoid and $X \subset \operatorname{Gpr} H$. Assume that there exists $\mathrm{GCD}(X)$. Then $\mathrm{LCM}(X) \in \operatorname{Gpr} H$.*

Proof. Denote $l = \mathrm{LCM}(X)$. Consider any element $b \in H$ such that $l \mid b^n$ for some $n \in \mathbb{N}$. Since l is the least common multiple of set X, then for any $c \in X$ we have $c \mid l$. Then $c \mid b^n$. Because $c \in \operatorname{Gpr} H$, so $c \mid b$. Then $l \mid b$. □

3 Square-free elements in a quotient monoid and a group of fractions

Proposition 3.1. *Let H be a monoid. Then $\operatorname{Sqf} H \subset \operatorname{Sqf} G$, where G is a group of fractions of H.*

Proof. Since G is a group, then $\operatorname{Sqf} G = G$. If $H \subset G$ and $a \in \operatorname{Sqf} H$, then $a \in G = \operatorname{Sqf} G$. □

Proposition 3.2. *Let H be a monoid. Then $\operatorname{Sqf} H/I \subset \operatorname{Sqf} H$ for some ideal I of H.*

Proof. Assume $a \in \operatorname{Sqf} H/I$ and suppose $a \notin \operatorname{Sqf} H$. Then $a = b^2c$, where $b \in H \setminus H^*$, $c \in H$. Since $a \notin H^*$, then $a \in I$ for some ideal I of H. Hence $a + I = b^2c + I = 0 + I$. A contradiction. □

Proposition 3.3. *Let H be a monoid. Then $\operatorname{Sqf} H \subset \operatorname{Sqf} H[X]$.*

Proof. Suppose that $s \notin \operatorname{Sqf} H[X]$. Then $s = f^2g$, where $f \in H[X] \setminus H$, $g \in H[X]$. Hence $s \in H[X] \setminus H$, so $s \notin \operatorname{Sqf} H$. □

4 Types of square-free and radical factorizations and conditions for the existence of some divisors

In this chapter we consider the relationship between square-free and radical factorizations and the conditions for the existence of some square-free or radical divisors in some monoids.

The following properties of the monoid H are paired: the square-free version and the radical version, for example in (0s)/(0r) the fragment ,,$s_1, s_2, \dots, s_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$" we read that for property (0s) we have ,,$s_1, s_2, \dots, s_n \in \operatorname{Sqf} H$", and for property (0r) we have "$s_1, s_2, \dots, s_n \in \operatorname{Gpr} H$".

Let H be a monoid. Consider the following conditions:

(0s) / (0r) for every $a \in H$ there exist $n \in \mathbb{N}$ and $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$ such that

$$a = s_1 s_2 \dots s_n,$$

(1s) / (1r) for every $a \in H$ there exist $n \in \mathbb{N}$ and $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$ satysfying the condition $s_i \operatorname{rpr} s_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$ such that

$$a = s_1 s_2^2 s_3^3 \dots s_n^n,$$

(2s) / (2r) for every $a \in H$ there exist $n \in \mathbb{N}$ and $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$ satysfying the condition $s_i \mid s_{i+1}$ for $i = 1, \dots, n-1$ such that

$$a = s_1 s_2 \dots s_n,$$

(3s) / (3r) for every $a \in H$ there exist $n \in \mathbb{N}_0$ and $s_0, s_1, \dots, s_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$ such that

$$a = s_0 s_1^2 s_2^{2^2} \dots s_n^{2^n},$$

(4s) / (4r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H / \operatorname{Gpr} H$ satysfying the condition $b \operatorname{rpr} c$ such that

$$a = bc$$

and there exists $d \in \operatorname{Sqf} H / \operatorname{Gpr} H$ such that $d^2 \mid b$ and $b \mid d^n$ for some $n \in \mathbb{N}$,

(4.1s) / (4.1r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H / \operatorname{Gpr} H$ satysfying the condition $b \operatorname{rpr} c$ such that

$$a = bc$$

and for every $d \in \operatorname{Sqf} H/ \operatorname{Gpr} H$, if $d \mid b$ then $d^2 \mid b$,

(4.2s) / (4.2r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ satysfying the condition $b \operatorname{rpr} c$ such that

$$a = bc$$

and there exists $d \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that $d^2 \mid b$,

(5s) / (5r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that

$$a = bc$$

and $a \mid c^n$ for some $n \in \mathbb{N}$,

(5.1s) / (5.1r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that

$$a = bc$$

and for every $d \in \operatorname{Sqf} H/ \operatorname{Gpr} H$, if $d \mid a$ then $d \mid c$,

(5.2s) / (5.2r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that

$$a = bc$$

and for every $d \in \operatorname{Sqf} H/ \operatorname{Gpr} H$, if $d \mid b$ then $d \mid c$,

(5.3s) / (5.3r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that

$$a = bc$$

and there exists $d \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that $d \mid c^n$ for some $n \in \mathbb{N}$,

(6s) / (6r) for every $a \in H$ there exist $b \in H$ and $c \in \operatorname{Sqf} H/ \operatorname{Gpr} H$ such that

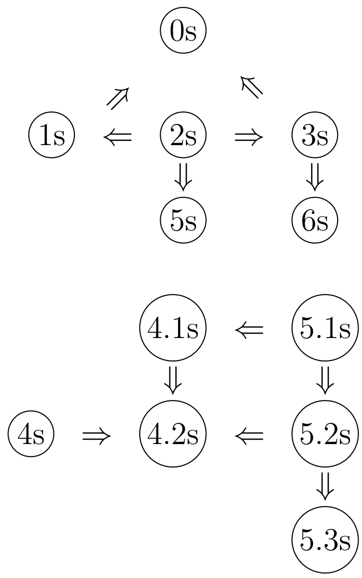
$$a = b^2 c.$$

5 Relationships between square-free and radical factorizations

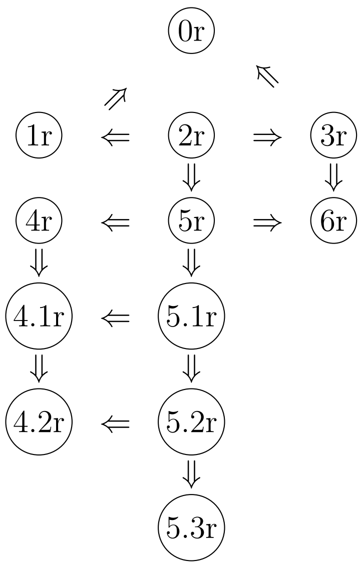
In this section we consider relationships between square-free and radical factorizations and conditions for the existence of some divisors.

Proposition 5.1. *Let H be a monoid.*

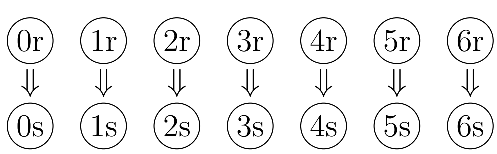
(a) The following implications holds:



(b) The following implications holds:



(c) The following implications holds:



Proof. **(a)**

(2s)⇒(1s) Put $t_n = s_1$. Since $s_1 \mid s_2$, then $s_2 = s_1 t_{n-1} = t_n t_{n-1}$ for some $t_{n-1} \in H$. Since $s_2 \mid s_3$, then $s_3 = s_2 t_{n-2} = t_n t_{n-1} t_{n-2}$ for some $t_{n-2} \in H$. Generally, we get $s_i = s_{i-1} t_{n-i+1} = t_n t_{n-1} \dots t_{n-i+1}$, where $t_n, t_{n-1}, \dots, t_{n-i+1} \in H$ for $i = 2, 3, \dots, n$. Hence

$$a = s_1 s_2 \dots s_n = t_n (t_n t_{n-1})(t_n t_{n-1} t_{n-2}) \dots (t_n t_{n-1} t_{n-2} \dots t_2 t_1) = \\ = t_n^n t_{n-1}^{n-1} t_{n-2}^{n-2} \dots t_2^2 t_1.$$

Since $s_n = t_n t_{n-1} \dots t_2 t_1$, then from Lemma 2.1 we refer $t_1, t_2, \dots, t_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$. While from Lemma 2.2 we get $t_i \operatorname{rpr} t_j$ for $i \neq j$.

(2s)⇒(3s) From (2s)⇒(1s) we can present an element a in the form $a = u_1 u_2^2 u_3^3 \dots u_n^n$, where $u_1, u_2, \dots, u_n \in \operatorname{Sqf} H / \operatorname{Gpr} H$ satysfying the condition $u_i \operatorname{rpr} u_j$ for $i, j \in \{1, 2, \dots , n\}$, $i \neq j$, where $s_{n-i+1} = s_{n-i} u_i$ for $i \in \{1, 2, \dots, n-1\}$ and $u_n = s_1$. Then

$$\prod_{k=1}^{n} u_k^k = \prod_{k=1}^{n} u_k^{\sum_{i=0}^{r} c_i^{(k)} 2^i} = \prod_{k=1}^{n} \prod_{i=0}^{r} u_k^{c_i^{(k)} 2^i} = \prod_{i=0}^{r} \Big(\prod_{k=1}^{n} u_k^{c_i^{(k)}} \Big)^{2^i}.$$

Denote $t_i = \prod_{k=1}^{n} u_k^{c_i^{(k)}}$ for $i = 0, 1, \dots r$. Because $u_i \operatorname{rpr} u_j$ for $i \neq j$, so from Lemma 2.2 we have $t_i \in \operatorname{Sqf} H$. Therefore $a = t_0 t_1^2 t_2^{2^2} \dots t_r^{2^r}$, where $k = \sum_{i=0}^{r} c_i^{(k)} 2^i$ for $k = 1, 2, \dots, n$ and $c_i^{(k)} \in \{0, 1\}$.

(2s)⇒(5s) Because

$$s_1 \mid s_n, s_2 \mid s_n, \dots, s_{n-1} \mid s_n, s_n \mid s_n,$$

hence $s_1 s_2 \dots s_n \mid s_n^n$, so $a \mid c^n$ for $c = s_n$. Of course $a = bc$, where $b = s_1 s_2 \dots s_{n-1}$.

(3s)⇒(6s), (4s)⇒(4.2s), (4.1s)⇒(4.2s) – Obvious.

(5.1s)⇒(4.1s) Let $a = bc$, where $b \in H$, $c \in \operatorname{Sqf} H$, $b \operatorname{rpr} c$ and let $d \in \operatorname{Sqf} H$ such that $d \mid b$. Since $d \mid b$, then $d \mid a$. By assumption we have $d \mid c$. Since $d \mid b$, $d \mid c$, then $d \in H^*$ because $b \operatorname{rpr} c$. Since $d \in H^*$ and $d \mid b$, then $d^2 \mid b$.

(5.2s)⇒(4.2s) Since $a = bc$, where $b \in H$, $c \in \operatorname{Sqf} H$ such that $b \operatorname{rpr} c$, and $d \mid b \Rightarrow d \mid c$, we get $d \in H^*$, and then $d^2 \mid b$.

(5.1s)⇒(5.2s) Let $d \in \operatorname{Sqf} H$ such that $d \mid b$. Then $d \mid a$ and by assumption we get $d \mid c$.

(5.2s)⇒(5.3s) Let $a = bc$, where $b \in H$, $c \in \operatorname{Sqf} H$. Let $d \in \operatorname{Sqf} H$ satisfies an implication $d \mid b \Rightarrow d \mid c$. Then $\prod_{\substack{d \in \operatorname{Sqf} H \\ d \mid b}}$ for some $n \in \mathbb{N}$. There exists $d' \in \operatorname{Sqf} H$ satisfies $d' \mid b$ such that $d' \mid \prod_{\substack{d \in \operatorname{Sqf} H \\ d \mid b}} d \mid c^n$. We get $d' \mid c^n$.

(b)

(4r)⇒(4.1r) Let $e \in \operatorname{Gpr} H$ be such that $e \mid b$. By assumption we have $b \mid d^n$, hence $e \mid d^n$, because $e \mid b$. But $e \in \operatorname{Gpr} H$, so from the fact that $e \mid d^n$ we have $e \mid d$, thus $e^2 \mid d^2$. By assumption we have $d^2 \mid b$, so $e^2 \mid b$.

(5r)⇒(4r) Let $a = bc$, where $b \in H, c \in \operatorname{Gpr} H$ such that $a \mid c^m$ for some $m \in \mathbb{N}$. By assumption we can b presented in the form $b = de$, where $d \in H$, $e \in \operatorname{Gpr} H$ such that $b \mid e^k$ for some $k \in \mathbb{N}$.

Since $e \mid b$, $b \mid a$ and $a \mid c^m$, then $e \mid c^m$. But $e \in \operatorname{Gpr} H$, so $e \mid c$ by definition. Then $c = ef$, where $f \in H$. By Lemma 2.1 we refer that $f \in \operatorname{Gpr} H$, and from Lemma 2.2 we have $e \operatorname{rpr} f$. From equation $b = de$ we have $be = de^2$. We get $a = bef$, where $e^2 \mid be$ and $be \mid e^{k+1}$.

Now we will prove that $be \operatorname{rpr} f$. From divisibilities $d \mid be$, $be \mid e^{k+1}$ and $e^{k+1} \mid c^{k+1}$ we have $d \mid c^{k+1}$ and $f \mid c$, $c \mid c^{k+1}$, so $f \mid c^{k+1}$. In other hand we have $df \mid bef$, $bef \mid a$ and $a \mid c^l$ for some $l \in \mathbb{N}$, so $df \mid c^l$. Hence since $d \mid c^k$, $f \mid c^l$, $df \mid c^l$, then $d \operatorname{rpr} f$. And since $e \operatorname{rpr} f$, then $be \operatorname{rpr} f$.

(5r)⇒(5.1r) Let $d \in \operatorname{Gpr} H$ be such that $d \mid a$. Since $d \mid a$ and by assumption $a \mid c^n$, then $d \mid c^n$. Because $d \in \operatorname{Gpr} H$, so $d \mid c$.

(5r)⇒(6r) Let $a = bc$, where $b \in H$, $c \in \operatorname{Gpr} H$. Since $b \mid a$ and $a \mid c^n$ for some $n \in \mathbb{N}$, then $c^n = bc'$ for some $c' \in H$. Hence $c' \mid c^n$. Since $c' \mid c^n$, then $c' \mid c$, i.e. $c = c'c''$, $c' \operatorname{rpr} c''$, where $c'' \in \operatorname{Sqf} H$.

We have $a^n = b^n c^n = b^n bc' = b^{n+1}c'$ and $a^n = b^n c^n = b^n c' n c''^n$. Then $b = c'^{n-1}c''^n$. We get $a = bc = c' n c''^{n+1} = (c'c'')^n c''$.

If $n = 2k$, then $a = (c'^k c''^k)^2 c''$. If $n = 2k+1$, then $a = (c'^k c''^{k+1})^2 c'$.

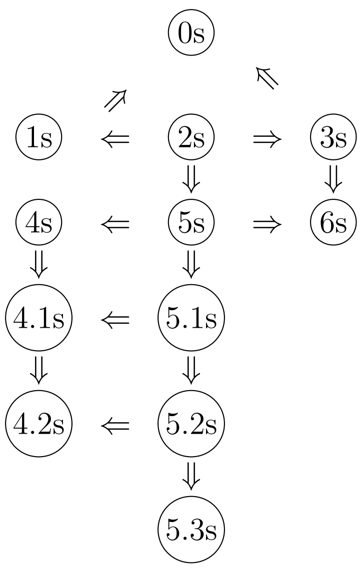
(c) The proof comes from the fact that every radical generator is a square-free.

□

Let's define another class of monoid. A monoid H is called SR-monoid, if $\operatorname{Gpr} H = \operatorname{Sqf} H$. Therefore it is enough to consider square-free properties.

Proposition 5.2. *Let H be a SR-monoid. Then*

(a) the following implications hold:



(b) the following equivalences hold:

$$\text{(Ar)} \Leftrightarrow \text{(As)}$$

for $A \in \{0, 1, 2, 3, 4, 4.1, 4.2, 5, 5.1, 5.2, 5.3, 6\}$.

Proof. **(a)** Since H is a SR-monoid, so every implications from Proposition 5.1 (b) hold.

(b) Obvious.

□

Since in pre-Schreier monoids, GCD-monoids and GCDs-monoids the SR property holds, therefore in the following three Propositions it is enough to consider square-free dependencies.

Proposition 5.3. *Let H be a pre-Schreier monoid. Then*

(a) the following implications and equivalences hold:

$$
\begin{array}{ccccc}
 & & \text{(0s)} & & \\
 & \Nearrow & & \Nwarrow & \\
\text{(1s)} & \Leftrightarrow & \text{(2s)} & \Rightarrow & \text{(3s)} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{(4s)} & \Leftrightarrow & \text{(5s)} & \Rightarrow & \text{(6s)} \\
\Downarrow & & \Downarrow & & \\
\text{(4.1s)} & \Leftrightarrow & \text{(5.1s)} & & \\
\Downarrow & & \Downarrow & & \\
\text{(4.2s)} & \Leftrightarrow & \text{(5.2s)} & & \\
 & & \Downarrow & & \\
 & & \text{(5.3s)} & &
\end{array}
$$

(b) if the condition (2s) *holds, then H be GCD-monoid.*

Proof. **(a)**

(1s)$\Rightarrow$(2s) An element a can be written in the form

$$a = s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\dots(s_2s_3\dots s_n)(s_1s_2s_3\dots s_n).$$

Put: $t_i = s_{n-i+1}s_{n-i+2}\dots s_{n-1}s_n$ for $i = 1, 2, \dots, n$. Because $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H$ and $s_i \operatorname{rpr} s_j$ for $i \neq j$ so from Lemma 2.3 (e) we have $t_1, t_2, \dots, t_n \in \operatorname{Sqf} H$. Of course $t_i \mid t_{i+1}$ for $i = 1, 2, \dots, n-1$.

(1s)$\Rightarrow$(4s) Put $b = s_2^2s_3^3\dots s_n^n$ and $c = s_1$. From the fact that $s_1, s_2, \dots, s_n$ are pairwise relatively prime results $b \operatorname{rpr} c$ from Lemma 2.3 (d). Moreover for $d = s_2s_3\dots s_n$ we have $d^2 \mid b, b \mid d^n$. Because $s_i \operatorname{rpr} s_j$ for $i, j \in \{2, 3, \dots, n\}$, $i \neq j$, so from Lemma 2.3 (e) we have $d \in \operatorname{Sqf} H$.

(4s)$\Rightarrow$(5s) Assume $a = bc$, where $b \in H$, $c \in \operatorname{Sqf} H$ such that $b \operatorname{rpr} c$ and $b = d^2e$, $b \mid d^m$, where $d \in \operatorname{Sqf} H$ and $m \in \mathbb{N}$. Then $a = d^2ec = (de)(cd)$. Since $d \mid b, b \operatorname{rpr} c$, then $d \operatorname{rpr} c$, so $cd \in \operatorname{Sqf} H$ by Lemma 2.3 (d). We get also that since $b \mid d^m$, then $bc \mid d^mc$, and because $d^mc \mid (cd)^m$, so $a \mid (cd)^m$.

(4.1s)$\Rightarrow$(5.1s) Let $a = bc$, where $b \in H$, $c \in \operatorname{Sqf} H$ such that $b \operatorname{rpr} c$. Let $d \in \operatorname{Sqf} H$ such that $d \mid b$. By assumption $d^2 \mid b$, i.e. $b = d^2g$ for some $g \in H$. Hence $a = bc = d^2gc = ef$, where $e = dg$, $f = dc$. Since $d \mid b$, $b \operatorname{rpr} c$, then $d \operatorname{rpr} c$.

By Lemma 2.3 (d) we have $cd = f \in \operatorname{Sqf} H$. Let $d' \in \operatorname{Sqf} H$ such that $d' \mid a$. Then $d' \mid ef$ and hence $d' \mid e$ or $d' \mid f$.

If $d' \mid f$, then end of proof. If $d' \mid e$, then $d' \mid dg$ and hence $d' \mid d^2g$, then $d' \mid b$. Since $d' \mid b$ and $b \operatorname{rpr} c$, then $d' \nmid c$, $d' \mid d$. We get $d' \mid cd$, so $d' \mid f$.

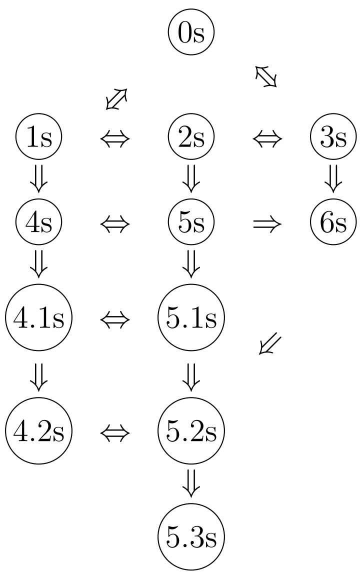
(4.2s)$\Rightarrow$(5.2s) Let $d \in \operatorname{Sqf} H$. We have $b = d^2g$ for some $g \in H$. We get $a = bc = d^2gc = ef$, $e = dg$, $f = dc$. Since $d \mid b$, $b \operatorname{rpr} c$, then $d \operatorname{rpr} c$. Since c, $d \in \operatorname{Sqf} H$, $c \operatorname{rpr} d$, then by Lemma 2.3(d) we have $cd = f \in \operatorname{Sqf} H$. Since $d \mid dg$, then $d \mid dc$.

The other implications hold from Proposition 5.1.

(b) Since every square-free element of a pre-Schreier monoid is radical, it follows from [5], Corollary 4.5, that every pre-Schreier monoid that satisfies property (2s) has to be a GCD-monoid. Note that the notion of a GCD-monoid is equivalent to the notion of a t-Bézout monoid in [5]. Therefore, if H is a pre-Schreier monoid that satisfy property (2s), then every principal ideal of H is a product of finitely many pairwise comparable radical principal ideals of H, and hence H is a t-Bézout monoid (i.e., a GCD-monoid) by [5], Corollary 4.5.

□

Proposition 5.4. *Let H be a GCD-monoid. Then the following implications and equivalences hold:*



Proof. (0s)$\Rightarrow$(2s) Since every square-free element in the GCD-monoid is radical and any element $a \in H$ can be presented as $a = s_1 s_2 \ldots s_n$, where s_1, s_2,

$\ldots, s_n \in \operatorname{Sqf} H$, then from the Theorem 3.10 from [5] we conclude that in every radical factorial GCD-monoid any radical r-finitely generated r-ideal of H is principal. And from the Corollary 4.5 from [5] we get that every principal ideal is a product of a finite number of pairwise principal ideals.

(6s)$\Rightarrow$(4.2s) Let $a = b^2c$, where $b \in H$, $c \in \operatorname{Sqf} H$. Let $d = \operatorname{GCD}(b, c)$. We have $b = db'$, $c = dc'$, where c', $d \in \operatorname{Sqf} H$, $b' \operatorname{rpr} c'$, $d \operatorname{rpr} c'$. We get $a = (d^3b'^2)c'$ and $d^2 \mid d^3b'^2$.

The other implications and equivalences hold from Proposition 5.3.

□

Proposition 5.5. *Let H be a GCDs-monoid. Then the condition* (5.1s) *holds.*

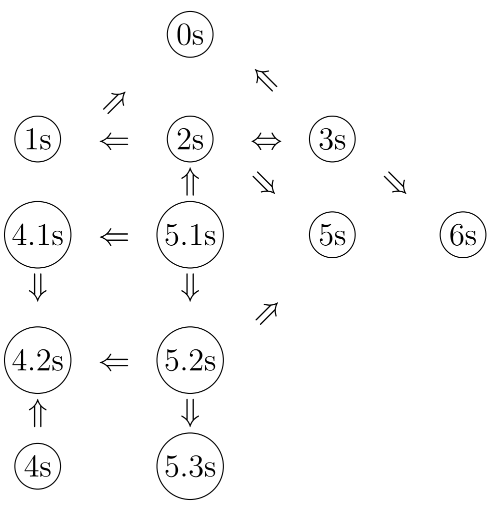
Proof. Let $a \in H$ and $X = \{d \in \operatorname{Sqf} H; d \mid a\}$. From Lemma 2.5 there exists $\operatorname{LCM}(X)$. Let $c = \operatorname{LCM}(X)$. By Lemma 2.6 we get that $c \in \operatorname{Sqf} H$. Since every element belonging to X divides a, the $c \mid a$. Hence $a = bc$ for some $b \in H$. Consider any $d \in \operatorname{Sqf} H$ such that $d \mid a$. But $d \in X$, hence $d \mid c$, because $c = \operatorname{LCM}(X)$. □

Note that in an atomic monoid the (0s) property holds and an implication (5s) $\Rightarrow$ (5.3s) holds. Indeed, let $a = q_1q_2 \ldots q_n$, where $q_i \in \operatorname{Irr} H \subset \operatorname{Sqf} H$. Then for some i we have $q_i \mid a$, $a \mid c^n$. Then $q_i \mid c^n$.

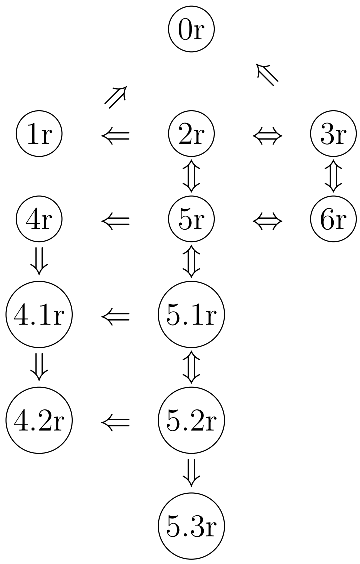
Proposition 5.6. *Let H be an ACCP-monoid. Then*

(a) the conditions (0s), (3s) *and* (6s) *hold,*

(b) the following implicatios and equivalences hold:



(c) the following implicatios and equivalences hold:



Proof. **(a)**

Any ACCP-monoid is atomic. Then (0s) holds.

If $a \in \operatorname{Sqf} H$, then put $c = a$ and $b = 1$.

Now, assume $a \notin \operatorname{Sqf} H$. Then $a = b_1^2 c_1$, where $b_1 \in H \setminus H^*$, $c_1 \in H$. If $c_1 \in \operatorname{Sqf} H$, then put $b = b_1$, $c = c_1$.

Now, assume $c_1 \notin \operatorname{Sqf} H$. Then $c_1 = b_2^2 c_2$, where $b_2 \in H \setminus H^*$, $c_2 \in H$.

We continue this process until $c_k \in \operatorname{Sqf} H$ for some $k \in \mathbb{N}$. Then

$$a = b_1^2 c_1 = b_1^2 b_2^2 c_2 = b_1^2 b_2^2 b_3^2 c_3 = \cdots = (b_1 b_2 \ldots b_k)^2 c_k.$$

Of course, this process has to stop. (6s) holds.

Consider any element $a \in H$. Since (6s) holds, then there exist $b_1 \in H$, $c_1 \in \operatorname{Sqf} H$ such that $a = b_1^2 c_1$. Again, the element b_1 can be written in the form $b_1 = b_2^2 c_2$, where $b_2 \in H$, $c_2 \in \operatorname{Sqf} H$. Similarly, we can introduce the element b_2 in the form $b_2 = b_3^2 c_3$, where $b_3 \in H$, $c_3 \in \operatorname{Sqf} H$. Continuing this reasoning, we obtain an ascending sequence of principal ideals ideals

$$(b_1) \subset (b_2) \subset (b_3) \subset \ldots$$

From ACCP-condition there exists $k \in \mathbb{N}$ such that

$$(b_k) = (b_{k+1}) = (b_{k+2}) = \ldots$$

In particular $(b_k) = (b_{k+1})$, i.e. $b_k \sim b_{k+1}$. Then from $b_k = b_{k+1}^2 c_{k+1}$ we refer $b_{k+1}, c_{k+1} \in H^*$. Since $b_k \sim b_{k+1}$ and $b_{k+1} \in H^*$, then $b_k \in H^*$.

Then

$$a = b_1^2 c_1 = b_2^{2^2} c_2^2 c_1 = b_3^{2^3} c_3^{2^2} c_2^2 c_1 = \dots = b_k^{2^k} c_1 c_2^2 c_3^{2^2} \dots c_k^{2^{k-1}} = s_0 s_1^2 s_2^{2^2} \dots s_n^{2^n},$$

where $s_0 = c_1$, $s_1 = c_2$, $s_2 = c^3$, ..., $s_{n-1} = c_k$, $s_n = b_k$. (3s) holds.

(b)

(5.1s)⇒(2s) Consider any element $a \in H$. We can presented element a in the form $a = b_1 c_1$, where $b_1 \in H$, $c_1 \in \operatorname{Sqf} H$ and for every $d \in \operatorname{Sqf} H$, if $d \mid a$, then $d \mid c$.

We can presented element b_1 in the form $b_1 = b_2 c_2$, where $b_2 \in H$, $c_2 \in \operatorname{Sqf} H$ and for every $d \in \operatorname{Sqf} H$, if $d \mid b_1$, then $d \mid c_2$.

An element b_2 we can presented in the form $b_2 = b_3 c_3$, where $b_3 \in H$, $c_3 \in \operatorname{Sqf} H$ and for every $d \in \operatorname{Sqf} H$, if $d \mid b_2$, then $d \mid c_3$.

Continuing, we get an ascending sequence of principal ideals

$$(b_1) \subset (b_2) \subset (b_3) \subset \dots .$$

Then by ACCP condition there exists $m \in \mathbb{N}$ such that

$$(b_n) = (b_{n+1}) = (b_{n+2}) \dots .$$

In particular $(b_k) = (b_{k+1})$, so $b_k \sim b_{k+1}$. Because $b_k = b_{k+1} c_{k+1}$, hence $c_{k+1} \in H^*$. we know that for any element $d \in \operatorname{Sqf} H$, if $d \mid b_k$, then $d \mid c_{k+1}$. But $c_{k+1} \in H^*$, hence since $d \mid b_k$, then $d \in H^*$.

We have

$$a = b_1 c_1 = b_2 c_2 c_1 = \dots = b_k c_k c_{k-1} \dots c_1 = c_k c_{k-1} \dots c_1,$$

because $b_k \in H^*$. We show that for every $i = 2, 3, \dots k$ the divisibiity $c_i \mid c_{i-1}$ holds. For $i = 2$ we have $c_2 \mid b_1$, because $b_1 = b_2 c_2$. Since $c_2 \mid b_1$, then $c_2 \mid a$. Then by the assumption $c_2 \mid c_1$. For $i = 3, 4, \dots$ we know that for every element b_{i-1} we can presented in the form $b_{i-1} = b_i c_i$, hence $c_i \mid b_{i-1}$. We also know that $b_{i-1} \mid b_{i-2}$. And hence $c_i \mid b_{i-2}$. By the assumption we have for any element $d \in \operatorname{Sqf} H$, if $d \mid b_{i-2}$, then $d \mid c_{i-1}$, so since $c_i \mid b_{i-2}$, then $c_i \mid c_{i-1}$, because $c_i \in \operatorname{Sqf} H$.

(6s)⇒(3s) Consider any element $a \in H$. The element a can be presented in the form $a = b_1^2 c_1$, where $b_1 \in H$, $c_1 \in \operatorname{Sqf} H / \operatorname{Gpr} H$.

An element b_1 can be presented in the form $b_1 = b_2^2 c_2$, where $b_2 \in H$, $c_2 \in \operatorname{Sqf} H/ \operatorname{Gpr} H$. Similarly, we can presented an element b_2 in the form $b_2 = b_3^2 c_3$, where $b_3 \in H$, $c_3 \in \operatorname{Sqf} H/ \operatorname{Gpr} H$.

By continuing this process, we obtain an ascending sequence of principal ideals

$$(b_1) \subset (b_2) \subset (b_3) \subset$$

By ACCP condition there exists $k \in \mathbb{N}$ such that $b_k \sim b_{k+1}$. And because $b_k = b_{k+1}^2 c_{k+1}$, hence $b_{k+1}, c_{k+1} \in H^*$. Since $b_k \sim b_{k+1}$ and $b_{k+1} \in H^*$, then $b_k \in H^*$.

Then

$$\begin{aligned} a &= b_1^2 c_1 = b_2^{2^2} c_2^2 c_1 = b_3^{2^3} c_3^{2^2} c_2^2 c_1 = \cdots = b_k^{2^k} c_k^{2^{k-1}} c_{k-1}^{2^{k-2}} \ldots c_2^2 c_1 = \\ &= s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n}, \end{aligned}$$

where $s_0 = c_1$, $s_1 = c_2$, $s_2 = c_3$, $\ldots$, $s_{n-1} = c_k$, $s_n = b_k$.

(5.2s)⇒(5s)

Consider any element $a \in H$. We can introduced the element a in the form $a = b_1 c_1$, where $b_1 \in H$, $c_1 \in \operatorname{Sqf} H$ and an implication $d \mid b_1 \Rightarrow d \mid c_1$ holds for every $d \in \operatorname{Sqf} H$.

An element b_1 can be presented in the form $b_1 = b_2 c_2$, where $b_2 \in H$, $c_2 \in \operatorname{Sqf} H$ and an implication $d \mid b_2 \Rightarrow d \mid c_2$ holds for every $d \in \operatorname{Sqf} H$. Next, we have $b_2 = b_3 c_3$, and so on.

For any $n \in \mathbb{N}$ we have an equation $b_n = b_{n+1} c_{n+1}$, where $b_{n+1} \in H$, $c_{n+1} \in \operatorname{Sqf} H$ and an implication $d \mid b_{n+1} \Rightarrow d \mid c_{n+1}$ holds.

By ACCP assumption we have $b_n c_n \sim b_{n+1} c_{n+1}$. Since $b_n = b_{n+1} c_{n+1}$, then $c_n \in H^*$. Since $d \mid b_n \Rightarrow d \mid c_n$ and $c_n \in H^*$, then $d \in H^*$. We get $\operatorname{Sqf} H = H^*$.

Since $a = b_n c_n c_{n-1} \ldots c_1$, then $a \mid c_1^n$.

(c)

(5r)⇒(2r)

Consider any element $a \in H$. We can introduced the element a in the form $a = b_1 c_1$, where $b_1 \in H$, $c_1 \in \operatorname{Gpr} H$ and $a \mid c_1^{n_1}$ holds for some $n_1 \in \mathbb{N}$.

An element b_1 can be presented in the form $b_1 = b_2 c_2$, where $b_2 \in H$, $c_2 \in \operatorname{Gpr} H$ and $b_1 \mid c_2^{n_2}$ holds form some $n_2 \in \mathbb{N}$.

An element b_2 can be presented in the form $b_2 = b_3 c_3$, where $b_3 \in H$, $c_3 \in \operatorname{Gpr} H$ and $b_2 \mid c_3^{n_3}$ holds for some $n_3 \in \mathbb{N}$.

Continuing our reasoning we get an increasing sequence of principal ideals

$$(b_1) \subset (b_2) \subset (b_3) \subset \dots$$

By ACCP condition there exists n such that

$$(b_n) = (b_{n+1}) = (b_{n+2}) = \dots .$$

In particular $(b_n) = (b_{n+1})$, so $b_n \sim b_{n+1}$. And because $b_n = b_{n+1}c_{n+1}$, so $c_{n+1} \in H^*$. There is also divisibility $b_n \mid c_{n+1}^{m_{n+1}}$, hence $b_n \in H^*$.

Then we get

$$a = b_1c_1 = b_2c_2c_1 = b_3c_3c_2c_1 = \dots = b_nc_nc_{n-1}\dots c_2c_1 = s_1s_2\dots s_n,$$

where $s_1 = b_nc_n$, $s_2 = c_{n-1}$, $s_3 = c_{n-2}$, $\dots$, $s_n = c_1$.

It remained to prove that for $i = 1, 2, \dots, n-1$ the condition $c_{i+1} \mid c_i$ holds. For $i = 1$ we have divisibilities $c_2 \mid b_1, b_1$, $b_1 \mid a$, $a \mid c_1^{m_1}$, hence $c_2 \mid c_1$, because $c_2 \in \operatorname{Gpr} H$. For $i > 1$ divisibilities $c_{i+1} \mid b_i$, $b_i \mid b_{i-1}$, $b_{i-1} \mid c_i^{m_i}$ holds, and hence $c_{i+1} \mid c_i$. Since $c_{i+1} \in \operatorname{Gpr} H$, then $c_{i+1} \mid c_i$.

(6r)⇒(5r) Consider any element $a \in H$. An element $a \in H$ can be presented in the form $a = b_1^2c_1$, where $b_1 \in H$, $c_1 \in \operatorname{Gpr} H$.

An element b_1c_1 can be presented in the form $b_1c_1 = b_2^2c_2$, where $b_2 \in H$, $c_2 \in \operatorname{Gpr} H$. Similarly, we can presented an element b_2c_2 in the form $b_2c_2 = b_3^2c_3$, where $b_3 \in H$, $c_3 \in \operatorname{Gpr} H$.

By repeating the process, we obtain the following ascending sequence of principal ideals

$$(b_1c_1) \subset (b_2c_2) \subset (b_3c_3) \subset \dots$$

By ACCP condition there exists $k \in \mathbb{N}$ such that

$$(b_kc_k) = (b_{k+1}c_{k+1}) = (b_{k+2}c_{k+2}) = \dots$$

In particular $(b_kc_k) = (b_{k+1}c_{k+1})$, so $b_kc_k \sim b_{k+1}c_{k+1}$. From the equation $b_kc_k = b_{k+1}^2c_{k+1}$ and from $b_kc_k \sim b_{k+1}c_{k+1}$ we get $b_{k+1} \in H^*$.

We have the following divisibility:

$$c_{k+1} \mid b_kc_k, b_kc_k \mid b_{k-1}c_{k-1}, \dots, b_2c_2 \mid b_1c_1, b_1c_1 \mid a.$$

Therefore, since $a = b_1^2c_1$, then $a \mid (b_1c_1)^2$. Since $b_1c_1 = b_2^2c_2$, then $b_1c_1 \mid (b_2c_2)^2$. Generally for $i = 2, 3, \dots, k$ we have $b_{k-1}c_{k-1} \mid (b_kc_k)^2$. Hence $a \mid (b_kc_k)^{2^k}$. Since $b_kc_k \sim c_{k+1}$, then $a \mid c_{k+1}^{2^k}$.

(5.2r)⇒(5r)

The proof is similar to (5.2s)⇒(5s) in (b).

The other implications hold from Proposition 5.1. □

6 Unique representation

In this section, we present the unique presentation of the factorizations and the conditions of existence of square-free and radical divisors.

Proposition 6.1. *Let H be a monoid.*

(a) *Consider any elements $s_1, s_2, \dots, s_n$, $t_1, t_2, \dots, t_n \in \operatorname{Gpr} H$, such that $s_i \mid s_{i+1}$ and $t_i \mid t_{i+1}$, for $i = 1, 2, \dots, n-1$. If*

$$r_1 r_2 \dots r_n \sim t_1 t_2 \dots t_n,$$

then $r_i \sim t_i$ for $i = 1, \dots, n$.

(b) *Consider any elements $a, c \in H$, $b, d \in \operatorname{Gpr} H$, such that $a \mid b^m$ and $c \mid d^n$ for some $m, n \in \mathbb{N}$. If*

$$ab \sim cd,$$

then $a \sim c$ and $b \sim d$.

(c) *Consider any elements $a, c \in H$, $b, d \in \operatorname{Gpr} H$, such that for any $e \in \operatorname{Gpr} H$ implications hold: if $e \mid ab$, then $e \mid b$ and if $e \mid cd$, then $e \mid d$. If*

$$ab \sim cd,$$

then $a \sim c$ and $b \sim d$.

Proof. **(a)** Assume $s_1 s_2 \dots s_n \sim t_1 t_2 \dots t_n$. From assumption we have:

$$t_1 \mid t_2,\ t_2 \mid t_3,\ \dots,\ t_{n-1} \mid t_n.$$

Then

$$t_1 \mid t_n,\ t_2 \mid t_n,\ \dots,\ t_{n-1} \mid t_n,\ t_n \mid t_n,$$

hence $t_1 t_2 \dots t_n \mid t_n^n$. Since $s_n \mid t_1 t_2 \dots t_n$, then $s_n \mid t_n^n$. Because $s_n \in \operatorname{Gpr} H$, then from definition we get $s_n \mid t_n$. We justify analogically $t_n \mid s_n$. Hence $s_n \sim t_n$ and then $s_1 \dots s_{n-1} \sim t_1 \dots t_{n-1}$.

Repeating the above reasoning for $s_1 \dots s_{n-1} \sim t_1 \dots t_{n-1}$ we get $s_{n-1} \sim t_{n-1}$ and $s_1 s_2 \dots s_{n-2} \sim t_1 t_2 \dots t_{n-2}$.

Continuing, we get $s_i \sim t_i$ for $i = 1, 2, \dots, n$.

b) Assume $ab \sim cd$. We notice $b \mid cd$, and since $c \mid d^n$, so $b \mid d^{n+1}$. Because $b \in \operatorname{Gpr} H$, then from definition we refer $b \mid d$. Similarly, we justify divisibility $d \mid b$. Therefore $b \sim d$, and then $a \sim c$.

(c) Assume $ab \sim cd$. We notice that $b \in \operatorname{Gpr} H$ and $b \mid cd$, so $b \mid d$ by assumption. Similarly, we justify divisibility $d \mid b$. Hence $b \sim d$, and then $a \sim c$.

□

Proposition 6.2. *Let H be a pre-Schreier monoid. Consider any elements $s_1, \dots, s_n$, $t_1, \dots, t_n \in \operatorname{Sqf} H$ such that $s_i \operatorname{rpr} s_j$ and $t_i \operatorname{rpr} t_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$. If*

$$s_1 s_2^2 s_3^3 \dots s_n^n \sim t_1 t_2^2 t_3^3 \dots t_n^n,$$

then $s_i \sim t_i$ for $i = 1, \dots, n$.

Proof. Assume $s_1 s_2^2 s_3^3 \dots s_n^n \sim t_1 t_2^2 t_3^3 \dots t_n^n$. Put $s_i' = s_i \dots s_n$, $t_i' = t_i \dots t_n$ for $i = 1, 2, \dots, n$. Then $s_1' s_2' \dots s_n' = s_1 s_2^2 s_3^3 \dots s_n^n$ and $t_1' t_2' \dots t_n' = t_1 t_2^2 t_3^3 \dots t_n^n$, so

$$s_1' s_2' \dots s_n' \sim t_1' t_2' \dots t_n'.$$

Because $s_i \operatorname{rpr} s_j$ and $t_i \operatorname{rpr} t_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, hence from Lemma 2.3 (e) we refer $s_i' = s_i s_{i+1} \dots s_n, t_i' = t_i t_{i+1} \dots t_n \in \operatorname{Sqf} H$ for $i = 1, 2, \dots, n$. Since $s_{i+1}' \mid s_i'$ and $t_{i+1}' \mid t_i'$ for $i = 1, 2, \dots, n-1$, then from Proposition 6.1 (a) we get $s_i' \sim t_i'$ for $i = 1, 2, \dots, n$, so $s_i s_{i+1} \dots s_n \sim t_i t_{i+1} \dots t_n$. Since $s_i s_{i+1} \dots s_n \sim t_i t_{i+1} \dots t_n$ and $s_{i+1} \dots s_n \sim t_{i+1} \dots t_n$, so $s_i \sim t_i$ for $i = 1, 2, \dots, n$. Moreover $s_n' \sim t_n'$, i.e. $s_n \sim t_n$. □

Proposition 6.3. *Let H be a GCD-monoid.*

(a) Consider any elements $a, c \in H$, $b, d \in \operatorname{Sqf} H$, such that $a \operatorname{rpr} b$, $c \operatorname{rpr} d$ and for some elements $e, f \in \operatorname{Sqf} H$ and $m, n \in \mathbb{N}$ divisibilities $e^2 \mid a$, $a \mid e^m$ and $f^2 \mid c$, $c \mid f^n$ hold. If

$$ab \sim cd,$$

then $a \sim c$, $b \sim d$.

(b) Consider any elements $a, c \in H$, $b, d \in \operatorname{Sqf} H$, such that $a \operatorname{rpr} b$, $c \operatorname{rpr} d$ and for any $g \in \operatorname{Sqf} H$ the implication holds: if $g \mid a$, then $g^2 \mid a$. If

$$ab \sim cd,$$

then $a \sim c$, $b \sim d$.

(c) Consider any elements $a, c \in H$ and $b, d \in \operatorname{Sqf} H$. If

$$a^2 b \sim c^2 d,$$

then $a \sim c$ and $b \sim d$.

(d) Consider any elements $s_0, s_1, \ldots, s_n \in \operatorname{Sqf} H$ *and* $t_0, t_1, \ldots, t_n \in \operatorname{Sqf} H$*. If*

$$s_n^{2^n} s_{n-1}^{2^{n-1}} \ldots s_1^2 s_0 \sim t_n^{2^n} t_{n-1}^{2^{n-1}} \ldots t_1^2 t_0,$$

then $s_i \sim t_i$ *for* $i = 0, 1, \ldots, n$*.*

Proof. **(a)** Assume $ab \sim cd$. Put $g = \mathrm{GCD}(d, e)$. Since $d \in \operatorname{Sqf} H$, then by Lemma 2.1 we have $g \in \operatorname{Sqf} H$, because $g \mid d$. Since $g \mid e$, then $g^2 \mid e^2$, and hence $g^2 \mid a$, because $e^2 \mid a$. Since $g^2 \mid a$ and $a \mid cd$, so $g^2 \mid cd$. Let us remind $g \mid d$, then $g^2 \mid d^2$. Since $g^2 \mid cd$, $g^2 \mid d^2$ and $c \operatorname{rpr} d$, hence by Lemma we refer $g^2 \mid \mathrm{GCD}(cd, d^2)$, so $g^2 \mid d$. Because $d \in \operatorname{Sqf} H$, so $g \in H^*$. Then $d \operatorname{rpr} e$, because g is their greatest common divisor. Therefore by Lemma 2.3 (c) we refer $d \operatorname{rpr} e^m$, and hence $d \operatorname{rpr} a$, because $a \mid e^m$. Similarly, we justify that $b \operatorname{rpr} c$ putting $h = \mathrm{GCD}(b, f)$ and we repeat the reasoning. Then by Lemma 2.3 (a) we have $a \sim c$, $b \sim d$.

(b) Assume $ab \sim cd$. Put $g = \mathrm{GCD}(a, d)$. Since $d \in \operatorname{Sqf} H$, then by Lemma 2.1 we have $g \in \operatorname{Sqf} H$, because $g \mid d$. Since $g \mid a$, then $g^2 \mid a$ by the assumption. Hence $g^2 \mid cd$. Let us remind $g \mid d$, then $g^2 \mid d^2$. Since $g^2 \mid cd$, $g^2 \mid d^2$ and $c \operatorname{rpr} d$, hence we refer $g^2 \mid \mathrm{GCD}(cd, d^2)$, so $g^2 \mid d$. Because $d \in \operatorname{Sqf} H$, so $g \in H^*$. Then $a \operatorname{rpr} d$, because g is their greatest common divisor. Because $d \mid ab$, hence $d \mid b$. Similarly, we justify that $b \operatorname{rpr} c$ putting $h = \mathrm{GCD}(b, c)$ and we repeat the reasoning. Then by Lemma 2.3 (a) we have $a \sim c$, $b \sim d$.

(c) Assume $a^2 b \sim c^2 d$. Put $e = \mathrm{GCD}(a, c)$ oraz $f = \mathrm{GCD}(b, d)$. Let $a = ea_0$, $c = ec_0$, where $a_0, c_0 \in H$ and $a_0 \operatorname{rpr} c_0$. Let $b = fb_0$ and $d = fd_0$, where $b_0, d_0 \in H$ and $b_0 \operatorname{rpr} d_0$.

We get $a^2 b = e^2 a_0^2 f b_0$ and $c^2 d = e^2 c_0 f d_0$. Since $a^2 b \sim c^2 d$, then $e^2 a_0^2 f b_0 \sim e^2 c_0 f d_0$, so $a_0^2 b_0 \sim c_0^2 d_0$. From Lemma 2.3 (e) since $a_0 \operatorname{rpr} c_0$, then $a_0^2 \operatorname{rpr} c_0^2$. We have $d_0 \mid a_0^2 b_0$ and $b_0 \operatorname{rpr} d_0$, so from Lemma 2.3 (a) we get $d_0 \mid a_0^2$. Similarly we have $a_0^2 \mid c_0^2 d_0$ and $a_0^2 \operatorname{rpr} c_0^2$, so from Lemma 2.3 (a) we get $a_0^2 \mid d_0$. Hence $a_0^2 \sim d_0$. We show analogously that $b_0 \sim c_0^2$.

Since $b_0 \mid b$, $d_0 \mid d$ and $b, d \in \operatorname{Sqf} H$, then from Lemma 2.1 we refer $b_0, d_0 \in \operatorname{Sqf} H$. But $b_0 \sim c_0^2$ and $d_0 \sim a_0^2$, so $b_0, d_0 \in \operatorname{Sqf} H$. And from $a_0^2 \sim d_0$, $c_0^2 \sim b_0$ we have $a_0, c_0 \in H^*$. Then we get $a \sim e$, $c \sim e$, so $a \sim c$. Analogously we get $b \sim f$, $d \sim f$, so $b \sim d$.

(d) Assume $s_n^{2^n} s_{n-1}^{2^{n-1}} \ldots s_1^2 s_0 \sim t_n^{2^n} t_{n-1}^{2^{n-1}} \ldots t_1^2 t_0$, where s_0, s_1, $\ldots$, s_n, t_0, t_1, $\ldots$, $t_n \in \operatorname{Sqf} H$.

Then $(s_1 s_2^2 \ldots s_n^{2^{n-1}})^2 s_0 \sim (t_1 t_2^2 \ldots t_n^{2^{n-1}})^2 t_0$. From (c) we get $s_0 \sim t_0$ and $s_1 s_2^2 \ldots s_n^{2^{n-1}} \sim t_1 t_2^2 \ldots t_n^{2^{n-1}}$.

Again using (c) for $s_1 s_2^2 \dots s_n^{2^{n-1}} = (s_2 s_3^2 \dots s_n^{2^{n-2}})^2 s_1 \sim t_1 t_2^2 \dots t_n^{2^{n-1}} = (t_2 t_3^2 \dots t_n^{2^{n-2}})^2 t_1$ we get $s_1 \sim t_1$ and $s_2 s_3^2 \dots s_n^{2^{n-2}} \sim t_2 t_3^2 \dots t_n^{2^{n-2}}$. By repeating the reasoning we get $s_i \sim t_i$ for $i = 0, 1, \dots, n$.

□

7 Some examples

Example 7.1. Let

$$H = \mathbb{N}_{\geqslant k} \cup \{0\}.$$

For any $k \in \mathbb{N}_0$ determine the set

$$H_k = \{(x, y) \in \mathbb{N}_0^2 \colon x + y = k\}.$$

For any $r \in \mathbb{N}$ consider the following submonoid of H:

$$H^{(r)} = \bigcup_{k \in \mathbb{N}_0} H_{kr}.$$

Then $H^{(r)}$ is a ACCP-monoid and

$\operatorname{Sqf} H^{(r)} = \{(0,0), (0,r), (1, r-1), \dots, (r-1, 1), (r, 0), (1, 2r-1), (3, 2r-3), \dots, (2r-1, 1)\}$.

For radical generators we have:
For $r = 1$ we have $\operatorname{Gpr} H^{(r)} = \{(0,0), (0,1), (1,0)\}$.
For $r = 2$ we have $\operatorname{Gpr} H^{(r)} = \{(0,0), (1,1)\}$.
For $r \geqslant 3$ we have $\operatorname{Gpr} H^{(r)} = \{(0,0)\}$.

For $r = 1$ All conditions are met.

For $r > 1$ the monoid $H^{(r)}$ conditions: $0s$, $1s$, $2s$, $3s$, $4s$, $5s$, $6s$, $4.1r$, $4.2r$, $5.1r$, $5.2r$, $5.3r$ are met. The other conditions are not met.

Example 7.2. For some $k \in \mathbb{N}$ let $H = \mathbb{Q}_{\geqslant k} \cup \{0\}$.

All quotient numbers of interval $[k, 2k)$ and 0 are square-free.

A monoid H is GCD-monoid. It sufficient put $\operatorname{GCD}(a, b) = \min\{a, b\}$ for all $a, b \in H$.

In the monoid H all conditions are met.

Example 7.3. Let $H = \mathbb{N}_0^2$. We have $H^* = \{(0,0)\}$, i.e. H is a reduced monoid.

For any $k \in \mathbb{N}_0$ determine a set

$$H_k = \{(x, y) \in \mathbb{N}_0^2 \colon x + y = k\}.$$

Then $H = \bigcup_{k\in\mathbb{N}_0} H_k$.

For any $r \in \mathbb{N}$ consider the following submonoid of H:

$$H^{(r)} = \bigcup_{k\in\mathbb{N}_0} H_{kr}.$$

Of course $H^{(r)}$ is a reduced monoid (as a submonoid of a reduced monoid) and

$$H^{(r)} = \langle (0, r), (1, r - 1), \dots , (r - 1, 1), (r, 0)\rangle.$$

For odd x and $k = 2$ elements $(x, kr - x)$ are square-free. Of course $(0, 0)$ also is square-free.

A submonoid of free-monoid is a monoid with finite factorial. In particular H is an ACCP-monoid.

In the monoid H conditions: 0s, 1s, 2s, 3s, 4s, 4.1r, 4.2r, 5s, 5.1r, 5.2r, 5.3r, 6s are met. The conditions 0r, 1r, 2r, 3r, 4r, 4.1s, 4.2r, 5r, 5.1s, 5.2s, 5.3s, 6r are not met.

Example 7.4. Let H be a monoid, not a group such that every element of H be a square. In particular $\mathbb{Q}_{\geqslant 0}$ and $\langle \frac{1}{2^n} \mid n \in \mathbb{N}\rangle$. In the monoid H the condition 6s is met. The others are not met.

Example 7.5. Consider a submonoid of free monoid

$$H = \langle x_1, x_2, \dots , y_1, y_2, \dots \mid y_i = x_{i+1}^p y_{i+1}^q, i = 1, 2, \dots\rangle,$$

where p, $q \in \mathbb{N}$. Then H is a non-factorial GCD-monoid for any p, q.

If $p = q = 1$, then in the H all conditions are met, in particular, it is non-atomic monoid satysfying 2s.

If q is even, then in the H 6s is met, and no one of 0s–5.3s and 0r–6r.

If q is odd and $(p, q) \neq (1, 1)$, then H satisfies no one of the conditions 0s/0r – 6s/6r.

8 The condition about square-free elements in a submonoid

In this section we introduce results about condition $\operatorname{Sqf} M \subset \operatorname{Sqf} H$, where M is a submonoid of the monoid H.

Let's recall that by $\mathcal{F}(B)$ we denote a free monoid with basis B, where $B \subset H$ is a subset.

Theorem 8.1. *Let H be a factorial monoid. Let $M \subset H$ be a submonoid such that $M^* = H^*$. The following conditions are equivalent:*

(a) $\operatorname{Sqf} M \subset \operatorname{Sqf} H$,

(b) $\operatorname{Irr} M \subset \operatorname{Sqf} H$ *and for every $a, b \in M$ the following implication holds:*

$$a \operatorname{rpr}_M b \Rightarrow a \operatorname{rpr}_H b,$$

(c) $\operatorname{Irr} M \subset \operatorname{Sqf} H$ *and for every $a, b \in \operatorname{Irr} M$ the following implication holds:*

$$a \not\sim_M b \Rightarrow a \operatorname{rpr}_H b,$$

(d) $M = H^* \times \mathcal{F}(B)$, *where B is an any set of pairwise relatively prime square-free non-units (of H),*

(e) *for every $s_1, s_2, \ldots, s_n \in \operatorname{Sqf} H$, such that $s_i \operatorname{rpr}_H s_j$ dla $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ the following implication holds:*

$$s_1 s_2^2 s_3^3 \ldots s_n^n \in M \Rightarrow s_1, s_2, \ldots, s_n \in M,$$

(f) *for every $q_1, q_2, \ldots, q_n \in \operatorname{Irr} M$ such that $q_i \not\sim_H q_j$ for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, the following implication holds:*

$$q_1^{k_1} \ldots q_n^{k_n} \in M \Rightarrow q_1^{c_i^{(1)}} \ldots q_n^{c_i^{(n)}} \in M,$$

where $k_j = c_r^{(j)} 2^r + \ldots + c_0^{(j)} 2^0$ for $j = 1, 2, \ldots, n$ with $c_i^{(j)} \in \{0, 1\}$ for $i = 0, 1, \ldots, r$,

(g) *for every $s_0, s_1, \ldots, s_n \in \operatorname{Sqf} H$ the following implication holds:*

$$s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} \in M \Rightarrow s_0, \ldots, s_n \in M,$$

(h) *for every $a \in H$ and $b \in \operatorname{Sqf} H$ the following implication holds:*

$$a^2 b \in M \Rightarrow a, b \in M.$$

Proof. First, we notice H is a BF-monoid (bounded factorization monoid), because is a factorial monoid. Submonoid M satisfies $M^* = H^* \cap M$, so M is also an BF-monoid ([7], Corollary 1.3.3, s. 17). In particular, M is an atomic monoid.

(a)⇒(c) Assume $\operatorname{Sqf} M \subset \operatorname{Sqf} H$. Since $\operatorname{Irr} M \subset \operatorname{Sqf} M$, then $\operatorname{Irr} M \subset \operatorname{Sqf} H$.

We show that if a, b are not relatively primes, then $a \nsim_M b$. Suppose that there exist $a, b \in \operatorname{Irr} M$ such that $a \nsim_M b$ and a, b are not relatively primes in H. Then $t = \operatorname{GCD}_H(a, b) \in H \setminus H^*$, so $a = tu$, $b = tv$ for some $u, v \in H$, $u \operatorname{rpr}_H v$. Since $a, b \in \operatorname{Irr} M$, then $a, b \in \operatorname{Sqf} H$, but $u \mid_H a$, $v \mid_H b$, so $u, v \in \operatorname{Sqf} H$, and then $uv \in \operatorname{Sqf} H$, because $u \operatorname{rpr}_H v$ (Lemma 2.3 (d))

We have $ab = t^2uv \notin \operatorname{Sqf} H$, because $t \in H \setminus H^*$. Hence by assumption $ab \notin \operatorname{Sqf} M$, i.e. $ab = c^2d$ for some $c \in M \setminus M^*$, $d \in M$. We can assume that $c \in M \setminus M^*$ is minimal and satisfies the following properties: "there are $a, b, d \in H$ such that $c \mid_H a, b$ and $ab = c^2d$ ". We have $c^2d = t^2uv$, where $uv \in \operatorname{Sqf} H$, so $c \mid_H t$, because H is factorial, and then $t = cw$ for some $w \in H$.

We get $a = tu = cwu$, so $uv \in \operatorname{Sqf} H$ because $a \in \operatorname{Sqf} H$. We have $ac = c^2wu \notin \operatorname{Sqf} H$, so $ac \notin \operatorname{Sqf} M$, hence $ac = e^2h$ for some $e \in M \setminus M^*$, $h \in M$. Since $e^2h = c^2wu$, where $wu \in \operatorname{Sqf} H$, we refer $e \mid_H c$, because $e^2h = c^2wu, a = cwu$, so $e^2h = ac$, and then $e \mid_H c$. Next, we have also $e \mid_H cwu$, then $e \mid_H a$. We get $e \mid_H a, c$ and $ac = e^2h$, so $e \sim_H c$ from minimal of c. Then $e \sim_M c$, because $M^* = H^*$. But $ac = e^2h$, so $a \sim_M eh \sim_M ch$. Then $a \sim_M c$ because $a \in \operatorname{Irr} M$ and $c \in M \setminus M^*$.

Similarly we show $b \sim_M c$, so $a \sim_M b$, a contradiction.

(b)⇒(c) It sufficient to notice that for every $a, b \in \operatorname{Irr} M$ the following implication holds

$$a \nsim_M b \Rightarrow a \operatorname{rpr}_M b.$$

Because, if a, $b \in \operatorname{Irr} M$ and $a \sim_M b$, then from (b) we have $a \operatorname{rpr}_H b$. Hence, if a, $b \in \operatorname{Irr} M$ are not relatively primes in M, then $a = cd$ and $b = ce$ for some $c \in M \setminus M^*$, $d, e \in M$. Because a, b are irreducible in M and c is non-ivertible, then $d, e \in M^*$. So, from $a = cd$, $b = ce$ we get $a \sim_M c$, $b \sim_M c$. We have $a \sim_M b$.

(c)⇒(b) Consider elements $a, b \in M$ such that $a \operatorname{rpr}_M b$. We know that M is an atomic monoid. Let $a = a_1 \dots a_m$ and $b = b_1 \dots b_n$ be factorizations to irreducible elements in M. Since $a \operatorname{rpr}_M b$, then for every i, j we have $a_i \nsim_M b_j$, so $a_i \operatorname{rpr}_H b_j$, but then $a \operatorname{rpr}_H b$.

(c)⇒(d) Let B be a maximal (with respect to inclusion) set of pairwise non-associative (in M) irreducibles in M. By (c) we refer that elements from

B are pairwise relatively prime in H. Since H is a factorial monoid, then B generates a free submonoid. Because M is atomic and $M^* = H^*$, then from [7] Theorem 1.2.3.2. we get $M = H^* \times B$.

(d)⇒(e) Let $a = s_1 s_2^2 s_3^3 \dots s_n^n \in M$, where $s_1, \dots, s_n \in \operatorname{Sqf} H$, $s_i \operatorname{rpr}_H s_j$ for $i \neq j$. By (d) an element a can be presented in the form $a = ct_1 t_2^2 t_3^3 \dots t_m^m$, where $c \in H^*$, $t_i = \prod_{j=1}^{r_i} b_j^{(i)} \in M$, $r_i \in \mathbb{N}_0$, $m \geq n$ and $b_j^{(i)} \in B$ such that $b_j^{(i)} \neq b_k^{(i)}$ for $j \neq k$. Since $b_j^{(i)}$ are square-free and pairwise relatively primes in H, then $t_1, \dots, t_m$ are also square-free, because in a factorial monoid the product of parwise relatively primes square-free is a square-free (Lemma 2.3 (d)). We refer that if $b_j^{(i)}$ are pairwise relatively primes, then $t_1, \dots, t_m$ are pairwise relatively primes in H. By Proposition 6.2 we get $s_i \sim_H t_i$ for $i = 1$, 2, …, n. Since $s_i \sim_H t_i$ and $t_i \in M$, then $s_i \in M$.

(e)⇒(f) Let $a = q_1^{k_1} \dots q_n^{k_n} \in M$, where $q_1, \dots, q_n \in \operatorname{Irr} H$, $q_i \not\sim_H q_j$ for $i \neq j$, and $k_1, \dots, k_n \in \mathbb{N}_0$. Put $m = \max(k_1, \dots, k_n)$. For $l = 1, \dots, m$ let's denote $s_l = \prod_{j\colon k_j = l} q_j$. Since $q_1, \dots, q_n \in \operatorname{Irr} H$, then $q_1, \dots, q_n \in \operatorname{Sqf} H$.

We show that since $q_i \not\sim_H q_j$ for $i \neq j$, then $q_i \operatorname{rpr}_H q_j$. Suppose that q_i, q_j are not relatively primes H, i.e. there exists $c \in H \setminus H^*$ and there exist $d, e \in H$ such that $q_i = cd$, $q_j = ce$. But $q_i, q_j \in \operatorname{Irr} H$, so $d, e \in H^*$. Hence $q_i \sim c$, $q_j \sim c$, so $q_i \sim q_j$.

Since $q_i \operatorname{rpr}_H q_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, then s_1, s_2, …, $s_m \in \operatorname{Sqf} H$. By definition s_l for $l = 1$, 2, …, m we get $s_i \operatorname{rpr}_H s_j$, where $i, j \in \{1, 2, \dots, n\}$, $i \neq j$. Then we have $a = s_1 s_2^2 \dots s_m^m$, so s_1, s_2, …, $s_m \in M$ by (e).

Now, let $k_j = c_r^{(j)} 2^r + \ldots + c_0^{(j)} 2^0$ for $j = 1$, 2, …, n, where $c_i^{(j)} \in \{0, 1\}$ for $i = 0, \dots, r$. We notice that if $k_{j_1} = k_{j_2}$, then $c_r^{(j_1)} 2^r + \ldots + c_0^{(j_1)} 2^0 = c_r^{(j_2)} 2^r + \ldots + c_0^{(j_2)} 2^0$, i.e. $c_i^{(j_1)} = c_i^{(j_2)}$ for some $i \in \{0, 1, \dots, r\}$. Let's denote $d_i^{(l)} = c_i^{(j)}$ for all j such that $k_j = l$, where $l = 1$, 2, …, m. Then $q_1^{c_i^{(1)}} \dots q_n^{c_i^{(n)}} = s_1^{d_i^{(1)}} \dots s_m^{d_i^{(m)}} \in M$.

(h)⇒(g) Let $s_0, \dots, s_n \in \operatorname{Sqf} H$ such that $s_0 s_1^2 s_2^{2^2} \dots s_n^{2^n} \in M$. By induction we prove, with respect to n, if $s_0 s_1^2 s_2^{2^2} \dots s_n^{2^n} \in M$, then $s_0, \dots, s_n \in M$. For $n = 1$ is obvious. Assume that this implication is true for any $n \in \mathbb{N}$. Let $s_0 s_1^2 s_2^{2^2} \dots s_n^{2^n} \in M$. Then $s_0 s_1^2 s_2^{2^2} \dots s_n^{2^n} = (s_1 s_2^2 s_3^{2^2} \dots s_n^{2^{n-1}})^2 s_0$. By (h) we get $s_1 s_2^2 s_3^{2^2} \dots s_n^{2^{n-1}}, s_0 \in M$.

Since $s_1 s_2^2 s_3^{2^2} \dots s_n^{2^{n-1}} \in M$, then $s_1 s_2^2 s_3^{2^2} \dots s_n^{2^{n-1}} = (s_2 s_3^2 s_4^{2^2} \dots s_n^{2^{n-2}})^2 s_1$. By (h) we have $s_2 s_3^2 s_4^{2^2} \dots s_n^{2^{n-2}}$, $s_1 \in M$.

Continuiyng this process we have s_0, s_1, …, $s_n \in M$.

(g)⇒(h) Consider $a \in H$, $b \in \operatorname{Sqf} H$. Let $a = s_n^{2^n} \dots s_1^2 s_0$, where $s_0, \dots, s_n \in$

Sqf H. If $a^2b = s_n^{2^{n+1}} \ldots s_1^{2^2} s_0^2 b \in M$, then $s_n, \ldots, s_1, s_0, b \in M$ by (g), and then $a = s_n^2 \ldots s_1^2 s_0 \in M$.

(f)⇒(g) Let $s_0, s_1, \ldots, s_n \in \operatorname{Sqf} H$. Assume $s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} \in M$. We can write $s_i = u_i q_1^{c_i^{(1)}} \ldots q_m^{c_i^{(m)}}$, where $u_i \in H^*$, $q_1, \ldots, q_m \in \operatorname{Irr} H$ such that $q_j \not\sim_H q_l$ for $j \neq l$ i $c_i^{(j)} \in \{0, 1\}$. Then $s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} = \prod_{i=0}^n (u_0 q_1^{c_0^{(1)}})^{2^i} = u_0 u_1^2 u_2^{2^2} \ldots u_n^{2^n} \cdot q_1^{k_1} \ldots q_m^{k_m}$, where $k_j = c_n^{(j)} 2^n + \ldots + c_1^{(j)} 2 + c_0^{(j)}$. By assumption, if $q_1^{k_1} \ldots q_m^{k_m} \in M$, then $q_1^{c_i^{(1)}} \ldots q_m^{c_i^{(m)}} \in M$ for $i = 1, 2, \ldots, n$. Hence $s_0, \ldots, s_n \in M$.

(g)⇒(f) Let $q_1^{k_1} \ldots q_n^{k_n} \in M$, where $q_1, \ldots, q_n \in \operatorname{Irr} H$, $q_j \not\sim_H q_l$ for $j, l \in \{1, 2, \ldots, n\}$, $j \neq l$. Put $k_j = c_r^{(j)} 2^r + \ldots + c_1^{(j)} 2 + c_0^{(j)}$ for $j = 1, 2, \ldots, n$, where $c_i^{(j)} \in \{0, 1\}$. Let $s_i = q_1^{c_i^{(1)}} \ldots q_n^{c_i^{(n)}}$. By assumption, since $s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} \in M$, then $s_0, s_1, \ldots, s_n \in M$. Hence $q_1^{c_i^{(1)}} \ldots q_n^{c_i^{(n)}} \in M$ for $i = 0, 1, \ldots, r$.

(h)⇒(a) Consider an element $r \in \operatorname{Sqf} R$. Suppose that $r \notin \operatorname{Sqf} H$, so $r = x^2 y$ for some $x, y \in H$ such that $x \notin H^*$ and $y \in \operatorname{Sqf} H$. Since $x^2 y \in M$, then we get $x, y \in M$. We have $x \notin M^*$, so $x^2 y \notin \operatorname{Sqf} M$, a contradiction. □

Let M be a submonoid of factorial monoid H. From Theorem 8.1 we know that the condition $\operatorname{Sqf} M \subset \operatorname{Sqf} H$ is equivalent to $\operatorname{Irr} M \subset \operatorname{Sqf} H$ and $a \operatorname{rpr}_M b \Rightarrow a \operatorname{rpr}_H b$ for every $a, b \in M$. Hence the condition $\operatorname{Irr} M \subset \operatorname{Sqf} H$ is equivalent to $\operatorname{Sqf} M \subset \operatorname{Sqf} H$, when for every $a, b \in M$ since $a \operatorname{rpr}_M b$ then $a \operatorname{rpr}_H b$.

9 The condition about irreducible elements in a submonoid

In this section we introduce results about condition $\operatorname{Irr} M \subset \operatorname{Sqf} H$, where M is a submonoid of the monoid H.

In Proposition 9.1 we find a factorial condition which implies the inclusion $\operatorname{Irr} M \subset \operatorname{Sqf} H$.

Proposition 9.1. *Let H be a monoid which satisfies the condition 6s. Let M be a submonoid of H. Assume that for every $a \in H$, $b \in \operatorname{Sqf} H$ the following implication holds*

$$a^2 b \in M \Rightarrow a, ab \in M.$$

Then $\operatorname{Irr} M \subset \operatorname{Sqf} H$.

Proof. Suppose that there exists some $c \in \operatorname{Irr} M$ such that $c \notin \operatorname{Sqf} H$. Then $c = a^2 b$ for some $a \in H$ and $b \in \operatorname{Sqf} H$. By assumption, since $a^2 b \in M$, then $a, ab \in M$. We notice $a \notin H^*$, because $c \notin \operatorname{Sqf} H$, so $a, ab \notin H^*$. And since $a, ab \notin H^*$, then a, $ab \notin M^*$ – a contradiciton with the assumption $a(ab) \in \operatorname{Irr} M$. □

An example 9.2 shows that in the Proposition 9.1 a factorial condition which implies $\operatorname{Irr} M \subset \operatorname{Sqf} H$ is not a necessary condition.

Example 9.2. Consider a monoid $H = \mathbb{N}_0^3$ and its submonoid

$$M = \langle (1,1,0), (1,0,1) \rangle.$$

Then $\operatorname{Irr} M = \{(1,1,0), (1,0,1)\}$, so $\operatorname{Irr} M \subset \operatorname{Sqf} H$, but for

$$a = (1,0,0) \in H,\ b = (0,1,1) \in \operatorname{Sqf} H$$

we have

$$2a + b = (2,0,0) + (0,1,1) = (2,1,1) = (1,1,0) + (1,0,1),$$

so $2a + b \in M$. Therefore $a \notin M$ and $a + b = (1,1,1) \notin M$.

A factorial condition which implies $\operatorname{Irr} M \subset \operatorname{Sqf} H$, i.e. for every $a \in H$, $b \in \operatorname{Sqf} H$ if $a^2 b \in M$ then a, $ab \in M$ is very interesting and we show next results for this factorial condition.

Theorem 9.3. *Let H be a factorial monoid. Let $M \subset H$ be a submonoid such that $M^* = H^*$. The following conditions are equivalent:*

(a) *for every $a \in H$ and $b \in \operatorname{Sqf} H$ the following implication holds*

$$a^2 b \in M \Rightarrow a, ab \in M,$$

(b) *for every $s_0, s_1, \ldots, s_n \in \operatorname{Sqf} H$, if*

$$s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} \in M,$$

then

$$s_n,\ s_{n-1}s_n,\ s_{n-2}s_{n-1}s_n^2,\ s_{n-3}s_{n-2}s_{n-1}^2 s_n^{2^2},\ \ldots,\ s_0 s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}} \in M,$$

(c) *for every $s_1, s_2, \ldots, s_n \in \operatorname{Sqf} H$ such that $s_i \operatorname{rpr}_H s_j$ for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, the implication holds*

$$s_1 s_2^2 s_3^3 \ldots s_n^n \in M \Rightarrow s_n, s_{n-1}s_n, s_{n-2}s_{n-1}s_n, \ldots, s_1 s_2 \ldots s_n \in M,$$

(d) *for every* $s_1,\ s_2,\ \ldots,\ s_n \in \operatorname{Sqf} H$, *such that* $s_i \mid s_{i+1}$ *for* $i = 1,\ 2,\ \ldots,\ n-1$, *the implication holds*

$$s_1 s_2 \ldots s_n \in M \Rightarrow s_1, s_2, \ldots, s_n \in M,$$

(e) *for every* $a \in H$ *and* $b \in \operatorname{Sqf} H$ *such that* $a \mid b^n$ *for some* $n \in \mathbb{N}$, *the implication holds*

$$ab \in M \Rightarrow a, b \in M.$$

Proof. (a)⇒(b) Consider elements $s_0, s_1, \ldots, s_n \in \operatorname{Sqf} H$ such that $s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} \in M$. Since $\left(s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}}\right)^2 s_0 \in M$, then from (a) we get:

$$s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}},\ \ \left(s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}}\right) s_0 \in M.$$

Next, since $\left(s_2 s_3^2 s_4^{2^2} \ldots s_n^{2^{n-2}}\right)^2 s_1 \in M$, then from (a) we get

$$s_2 s_3^2 s_4^{2^2} \ldots s_n^{2^{n-2}},\ \ \left(s_2 s_3^2 s_4^{2^2} \ldots s_n^{2^{n-2}}\right) s_1 \in M.$$

Continuing, since $\left(s_{n-1} s_n^2\right)^2 s_{n-2} \in M$, then from (a) we get:

$$s_{n-1} s_n^2, s_{n-1} s_2^2 s_{n-2} \in M.$$

Since $s_n^2 s_{n-1} \in M$, then from (a) we get $s_{n-1}, s_n \in M$.

From all steps we have:

$$\left(s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}}\right) s_0,\ \left(s_2 s_3^2 s_4^{2^2} \ldots s_n^{2^{n-2}}\right) s_1, \ldots,\ s_{n-1} s_n^2 s_{n-2},\ s_n s_{n-1},\ s_n \in M.$$

(b)⇒(a) Consider $a \in H$, $b \in \operatorname{Sqf} H$ such that $a^2 b \in M$. A monoid H is factorial, so an element a can be written in the form $a = s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}}$, where $s_i \in \operatorname{Sqf} H$ for $i = 1, \ldots, n$. Put $s_0 = b$. Then we get:

$$s_0 s_1^2 s_2^{2^2} \ldots s_n^{2^n} = a^2 b \in M.$$

By assumption we have

$$s_0 s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}},\ s_1 s_2 s_3^2 s_4^{2^2} \ldots s_n^{2^{n-2}}, \ldots,\ s_{n-2} s_{n-1} s_n^2,\ s_{n-1} s_n,\ s_n \in M.$$

Notice $ab = s_0 s_1 s_2^2 s_3^{2^2} \ldots s_n^{2^{n-1}}$, so $ab \in M$. Moreover

$$a = s_1 s_2^2 \ldots s_n^{2^{n-1}} =$$
$$= s_n (s_{n-1} s_n) \left(s_{n-2} s_{n-1} s_n^2\right) \left(s_{n-3} s_{n-2} s_{n-1}^2 s_n^{2^2}\right) \left(s_{n-4} s_{n-3} s_{n-2}^2 s_{n-1}^{2^2} s_n^{2^3}\right) \ldots$$
$$\left(s_2 s_3 s_4^2 s_5^{2^2} \ldots s_n^{2^{n-1}}\right) \left(s_1 s_2 s_3^2 s_4^{2^2} \ldots s_n^{2^{n-2}}\right),$$

so $a \in M$.

(b)$\Rightarrow$(c) Denote by $\lceil x \rceil$ and $\lfloor x \rfloor$ upper part (ceiling) and bottom part (floor) of real number x.

Step I. First, we prove that if $s_1 s_2^2 s_3^3 \dots s_n^n \in M$, where $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H$, $s_i \operatorname{rpr}_H s_j$ for $i \neq j$, then $s_1 s_2 s_3^2 s_4^2 \dots s_n^{\lceil \frac{n}{2} \rceil}$, $s_2 s_3 s_4^2 s_5^2 \dots s_n^{\lfloor \frac{n}{2} \rfloor} \in M$.

Let $a = s_1 s_2^2 s_3^3 \dots s_n^n \in M$, where $s_1, \dots, s_n \in \operatorname{Sqf} H$, $s_i \operatorname{rpr}_H s_j$ for $i \neq j$. Then an element a can be written in the form $a = t_0 t_1^2 t_2^{2^2} \dots t_r^{2^r}$, where $t_i = s_1^{c_i^{(1)}} \dots s_n^{c_i^{(n)}} \in \operatorname{Sqf} H$, $i = 0, \dots, r$ and $k = \sum_{i=0}^{r} c_i^{(k)} 2^i$, where $c_i^{(k)} \in \{0, 1\}$, $k = 0, 1, \dots, n$ (see proof of (2s)$\Rightarrow$(3s), Proposition 5.1).

From (b) we have

$$t_0 t_1 t_2^2 t_3^{2^2} \dots t_r^{2^{r-1}}\ t_1 t_2 t_3^2 t_4^{2^2} \dots t_r^{2^{r-2}}, \dots, t_{n-2} t_{n-1} t_n^2,\ t_{n-1} t_n,\ t_n \in M.$$

Then

$$\begin{aligned} t_1 t_2^2 t_3^{2^2} \dots t_r^{2^{r-1}} &= \\ &= (t_1 t_2 t_3^2 t_4^{2^2} \dots t_r^{2^{r-2}})(t_2 t_3 t_4^2 t_5^{2^2} \dots t_r^{2^{r-3}}) \dots (t_{r-2} t_{r-1} t_r^2)(t_{r-1} t_r) t_r \in M. \end{aligned}$$

From definition of exponents $c_i^{(j)}$ we have

$$s_1 s_2 s_3^2 s_4^2 \dots s_n^{\lceil \frac{n}{2} \rceil} = t_0 t_1 t_2^2 \dots t_r^{2^{r-1}} \in M,$$

$$s_2 s_3 s_4^2 s_5^2 \dots s_n^{\lfloor \frac{n}{2} \rfloor} = t_1 t_2^2 t_3^{2^2} \dots t_r^{2^{r-1}} \in M.$$

Step II. Now, we prove that if $s_1 s_2^2 s_3^3 \dots s_n^n \in M$, where $s_1, \dots, s_n \in \operatorname{Sqf} H$, $s_i \operatorname{rpr}_H s_j$ for $i \neq j$, then $s_1 s_2 s_3 \dots s_n$, $s_2 s_3^2 s_4^3 \dots s_n^{n-1} \in M$.

Assume $s_1 s_2^2 s_3^3 \dots s_n^n \in M$, where $s_1, \dots, s_n \in \operatorname{Sqf} H$, $s_i \operatorname{rpr}_H s_j$ for $i \neq j$. We prove by induction with respect to l that

$$s_1^{\lceil \frac{1}{2^l} \rceil} s_2^{\lceil \frac{2}{2^l} \rceil} \dots s_{n-1}^{\lceil \frac{n-1}{2^l} \rceil} s_n^{\lceil \frac{n}{2^l} \rceil},\quad s_1^{1 - \lceil \frac{1}{2^l} \rceil} s_2^{2 - \lceil \frac{2}{2^l} \rceil} \dots s_{n-1}^{n-1-\lceil \frac{n-1}{2^l} \rceil} s_n^{n - \lceil \frac{n}{2^l} \rceil} \in M.$$

Let $q = \lceil \frac{n}{2^l} \rceil$. Then $(q-1)2^l < n \leq q 2^l$. Put

$$s'_i = s_{(i-1)2^l+1} s_{(i-1)2^l+2} \dots s_{i2^l}$$

for $i = 1, \dots, q-1$ and $s'_q = s_{(q-1)2^l+1} s_{(q-1)2^l+2} \dots s_n$.

Notice $s'_1, s'_2, \dots, s'_q \in \operatorname{Sqf} H$ and $s'_i \operatorname{rpr}_H s'_j$ for $i \neq j$, because $s_1, \dots, s_n \in \operatorname{Sqf} H$, $s_i \operatorname{rpr}_H s_j$ for $i \neq j$.

We have

$$s_1^{\lceil\frac{1}{2^l}\rceil}s_2^{\lceil\frac{2}{2^l}\rceil}\dots s_{n-1}^{\lceil\frac{n-1}{2^l}\rceil}s_n^{\lceil\frac{n}{2^l}\rceil}=s'_1(s'_2)^2\dots(s'_q)^q.$$

If $s_1^{\lceil\frac{1}{2^l}\rceil}s_2^{\lceil\frac{2}{2^l}\rceil}\dots s_{n-1}^{\lceil\frac{n-1}{2^l}\rceil}s_n^{\lceil\frac{n}{2^l}\rceil}\in M$, then by Step I we have

$$s_1^{\lceil\frac{1}{2^{l+1}}\rceil}s_2^{\lceil\frac{2}{2^{l+1}}\rceil}\dots s_{n-1}^{\lceil\frac{n-1}{2^{l+1}}\rceil}s_n^{\lceil\frac{n}{2^{l+1}}\rceil}=s'_1s'_2(s'_3)^2(s'_4)^2\dots(s'_q)^{\lceil\frac{q}{2}\rceil}\in M$$

and

$$s_1^{\lceil\frac{1}{2^l}\rceil-\lceil\frac{1}{2^{l+1}}\rceil}s_2^{\lceil\frac{2}{2^l}\rceil-\lceil\frac{2}{2^{l+1}}\rceil}\dots s_n^{\lceil\frac{n}{2^l}\rceil-\lceil\frac{n}{2^{l+1}}\rceil}=s'_2s'_3(s'_4)^2(s'_5)^2\dots(s'_q)^{\lfloor\frac{q}{2}\rfloor}\in M.$$

Since $s_1^{1-\lceil\frac{1}{2^l}\rceil}s_2^{2-\lceil\frac{2}{2^l}\rceil}\dots s_n^{n-\lceil\frac{n}{2^l}\rceil}\in M$, then also

$$s_1^{1-\lceil\frac{1}{2^{l+1}}\rceil}s_2^{2-\lceil\frac{2}{2^{l+1}}\rceil}\dots s_n^{n-\lceil\frac{n}{2^{l+1}}\rceil}=$$

$$s_1^{1-\lceil\frac{1}{2^l}\rceil}s_2^{2-\lceil\frac{2}{2^l}\rceil}\dots s_n^{n-\lceil\frac{n}{2^l}\rceil}\cdot s_1^{\lceil\frac{1}{2^l}\rceil-\lceil\frac{1}{2^{l+1}}\rceil}s_2^{\lceil\frac{2}{2^l}\rceil-\lceil\frac{2}{2^{l+1}}\rceil}\dots s_n^{\lceil\frac{n}{2^l}\rceil-\lceil\frac{n}{2^{l+1}}\rceil}\in M.$$

There exists $r\in\mathbb{N}$ such that $2^r>n$. Then $1\le t\le n$ and we have $\lceil\frac{t}{2^r}\rceil=1$. Finally, we have $s_1s_2s_3\dots s_n,\ s_2s_3^2s_4^3\dots s_n^{n-1}\in M$.

Step III. Now, we prove (c) by induction with respect to $n\in\mathbb{N}$. For $n=1$ is obvious. Assumme that the condition holds for n and consider $s_1,s_2,\dots,$ $s_n,s_{n+1}\in\operatorname{Sqf}H$, $s_i\operatorname{rpr}_H s_j$ for $i\neq j$ such that $s_1s_2^2s_3^3\dots s_n^ns_{n+1}^{n+1}\in M$. By Step II we have

$$s_1s_2s_3\dots s_ns_{n+1},\ s_2s_3^2s_4^3\dots s_n^{n-1}s_{n+1}^n\in M.$$

By inductive assumption for element $s_2s_3^2s_4^3\dots s_n^{n-1}s_{n+1}^n$ we have

$$s_{n+1},\ s_ns_{n+1},\ s_{n-1}s_ns_{n+1},\ \dots,\ s_2s_3\dots s_ns_{n+1}\in M.$$

(c)⇒(b) We apply induction with respect to n. The case for $n=0$ is obvious.

Assume that the condition holds for all $n\in\mathbb{N}$, i.e., for every $s_0,s_1,\dots,s_n\in$ $\operatorname{Sqf}H$, if $s_0s_1^2s_2^{2^2}\dots s_n^{2^n}\in M$, then $s_{n-l}s_{n-l+1}s_{n-l+2}^2s_{n-l+3}^{2^2}\dots s_{n-1}^{2^{l-2}}s_n^{2^{l-1}}\in M$ for every $l\in\{0,1,\dots,n\}$.

We prove the condition for $n+1$. Let

$$a=s_0s_1^2s_2^{2^2}\dots s_{n+1}^{2^{n+1}}\in M,$$

where $s_0,s_1,\dots,s_{n+1}\in\operatorname{Sqf}H$. Then by Proposition 5.4, (3s)⇒(1s) element a can be written in the form

$$a=t_1t_2^2t_3^3\dots t_m^m,$$

where $m=2^{n+2}-1$ and $t_1,\dots,t_m\in\operatorname{Sqf}H$, $t_i\operatorname{rpr}_H t_j$ for $i\neq j$.

By (c) we have

$$t_m,\ t_{m-1}t_m,\ \dots,\ t_1t_2\dots t_m \in M.$$

Notice m is odd, because $m = 2^{n+2} - 1$. Multiplying elements in the form $t_rt_{r+1}\dots t_m$ for every odd r we get

$$(t_1t_2t_3t_4\dots t_m)(t_3t_4t_5\dots t_m)\dots(t_{m-2}t_{m-1}t_m)t_m = t_1t_2t_3^2t_4^2\dots t_m^{\lceil\frac{m}{2}\rceil} \in M.$$

And if we multiply elements in the form $t_rt_{r+1}\dots t_m$ for even r, then we get

$$(t_2t_3t_4t_5\dots t_m)(t_4t_5t_6\dots t_m)\dots(t_{m-1}t_m) = t_2t_3t_4^2t_5^2\dots t_m^{\lfloor\frac{m}{2}\rfloor} \in M.$$

Notice, since $m = 2^{n+2} - 1$, then $\lfloor\frac{m}{2}\rfloor = 2^{n+1} - 1$. Hence

$$t_2t_3t_4^2t_5^2\dots t_m^{\lfloor\frac{m}{2}\rfloor} = (t_2t_3)(t_4t_5)^2\dots t_m^{\lfloor\frac{m}{2}\rfloor},$$

because t_2t_3, t_4t_5, ..., $t_m \in \operatorname{Sqf} H$ by Lemma 2.3 (d). From proof of Proposition 5.4 (1s)⇔(3s) we get

$$(t_2t_3)(t_4t_5)^2\dots t_m^{\lfloor\frac{m}{2}\rfloor} = s_1s_2^2s_3^{2^2}\dots s_{n+1}^{2^n}.$$

By inductive assumption we have

$$s_{n-l}s_{n-l+1}s_{n-l+2}^2s_{n-l+3}^{2^2}\dots s_{n-1}^{2^{l-2}}s_n^{2^{l-1}} \in M$$

for $l \in \{0, 1, \dots, n\}$. Moreover,

$$t_1t_2t_3^2t_4^2\dots t_m^{\lceil\frac{m}{2}\rceil} = (t_1t_2)(t_3t_4)^2\dots t_m^{\lceil\frac{m}{2}\rceil},$$

because t_1t_2, t_3t_4, ..., $t_m \in \operatorname{Sqf} H$ by Lemma 2.3 (d). From proof of Proposition 5.4 (1s)⇔(3s) we get

$$(t_1t_2)(t_3t_4)^2\dots t_m^{\lceil\frac{m}{2}\rceil} = s_0s_1s_2^2s_3^{2^2}\dots s_{n+1}^{2^n},$$

i.e. the condition for $l = n + 1$.

(c)⇔(d) Since H is a factorial monoid, then by Proposition 5.3 we have that an element $a \in H$ in the form

$$a = s_1s_2^2s_3^3\dots s_n^n,$$

where $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H$ satisfy the condtion $s_i \operatorname{rpr} s_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, can be written in the form

$$a = t_1t_2t_3\dots t_n,$$

where elements $t_1, t_2, t_3, \dots, t_n \in \operatorname{Sqf} H$ satisfy the condition $t_i \mid t_{i+1}$ for $i = 1, 2, \dots, n-1$.

In othe hands, by Proposition 5.1 we refer that an element $a \in H$ in the form

$$a = t_1 t_2 t_3 \dots t_n,$$

where elements $t_1, t_2, t_3, \dots, t_n \in \operatorname{Sqf} H$ satisfy the condition $t_i \mid t_{i+1}$ for $i = 1, 2, \dots, n-1$, can be presented in the form

$$a = s_1 s_2^2 s_3^3 \dots s_n^n,$$

where elements $s_1, s_2, \dots, s_n \in \operatorname{Sqf} H$ satisfy the condition $s_i \operatorname{rpr} s_j$ for $i, j \in \{1, 2, \dots, n\}$, $i \neq j$.

(d)⇒(e) Consider $a \in H$, $b \in \operatorname{Sqf} H$ such that $a \mid b^n$ for some $n \in \mathbb{N}$ and $ab \in M$. Let $a = s_1 s_2 \dots s_m$, where $s_1, s_2, \dots, s_m \in \operatorname{Sqf} H$, $s_i \mid s_{i+1}$ for $i = 1, 2, \dots, m-1$.

Notice, for $i = 1, 2, \dots, m$ we have $s_i \mid a$. By assumption we have $a \mid b^n$, hence $s_i \mid b^n$. In particular $s_m \mid b^n$. Of course H is a factorial monoid, so $\operatorname{Sqf} H = \operatorname{Gpr} H$. Since $s_m \mid b^n$, then $s_m \mid b$, because $s_m \in \operatorname{Gpr} H$. We have $s_1 s_2 \dots s_m b = ab \in M$. By (d) we get $s_1, s_2, \dots, s_m, b \in M$, so $a, b \in M$.

(e)⇒(d) Let $s_1 s_2 \dots s_n \in M$, where $s_1, \dots, s_n \in \operatorname{Sqf} H$ satisfy the condition $s_i \mid s_{i+1}$ for $i = 1, \dots, n-1$.

Put $a = s_1 s_2 \dots s_{n-1}$, $b = s_n$. Then

$$s_1 \mid s_2,\ s_2 \mid s_3,\ \dots,\ s_{n-1} \mid s_n.$$

Hence $s_1 s_2 \dots s_{n-1} \mid s_n^{n-1}$, i.e. $a \mid b^{n-1}$. By (e) we have $s_1 s_2 \dots s_{n-1} \in M$ and $s_n \in M$.

Put $a = s_1 s_2 \dots s_{n-2}$, $b = s_{n-1}$. Then

$$s_1 \mid s_2,\ s_2 \mid s_3,\ \dots,\ s_{n-2} \mid s_{n-1}.$$

Hence $s_1 s_2 \dots s_{n-2} \mid s_n^{n-2}$, i.e. $a \mid b^{n-2}$. By (e) we have $s_1 s_2 \dots s_{n-2} \in M$ and $s_{n-1} \in M$.

Repeating this process we get $s_n,\ s_{n-1},\ s_{n-2},\ \dots,\ s_2,\ s_1 \in M$. □

In Proposition 9.1 we found the factorial condition: $a^2 b \in M \Rightarrow a, ab \in M$ for every $a \in H$, $b \in \operatorname{Sqf} H$, which implies the condition $\operatorname{Irr} M \subset \operatorname{Sqf} H$. But from Example 9.2 we know this condition is not necessary. That means the condition $\operatorname{Irr} M \subset \operatorname{Sqf} H$ it is generally not equivalent to factorial factorial condition but ,,behaves well” with respect to different square-free factorizations.

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