THE JACOBIAN CONJECTURE AND SQUARE-FREE IDEALS

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Abstract. In this paper we present an equivalent statement to the Jacobian conjecture using square-free and maximal ideals. We will show that the equivalent hypothesis is true, which implies that the Jacobian conjecture is true.

1. Introduction

The Jacobian conjecture, formulated by Keller [1] in 1939, is one of the most important open problems stimulating modern mathematical research (see [2]). In this article we deal with the problem of the Jacobian conjecture for $\mathbb{C}^n$. The results can be generalized to an n-dimensional algebraically closed field. We present a positive solution to this conjecture.

Jacobian conjecture. If the polynomial mapp $F\colon \mathbb{C}^n \to \mathbb{C}^n$ has a non-zero Jacobian constant, then F is an automorphism.

This conjecture is one of the classic problems of polynomial mapping theory and has many implications and applications in algebraic geometry, number theory, and holomorphic dynamics. There are various approaches to this problem, based on algebraic, analytical or combinatorial methods. More information on this subject can be found in two monographs [3], [5].

It is worth noting that in [4] the authors showed the relationship between the Jacobian hypothesis and irreducible and square-free elements in certain rings of polynomials. In this article, we will also show relationships, although not motivated by [4].

Article [6] defines the concept of a square-free ideal, i.e. it is an ideal I in the ring R, where it cannot be represented as $I = J^2K$, where J and K are ideals in R, different from I, and J is a proper ideal. Also in [6], equivalent

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definitions of the square-free ideal are presented, but in this article we will also use the definition that $I^2 = I$.

In this article, we will show an equivalent statement to the Jacobian hypothesis, which is based on the square-free and maximal ideals. We will also present a positive solution to the Jacobian hypothesis.

2. Main result

Let us begin by presenting an equivalent statement to the Jacobian conjecture.

Theorem 2.1. *Let $A = \mathbb{C}[x_1, \dots, x_n]$. Let $I = (F_1, \dots, F_n$ be the ideal generated by the coordinates $F = (F_1, \dots, F_n)\colon \mathbb{C}^n \to \mathbb{C}^n$. Let $J = (J_1, \dots, J_n)$ be the ideal generated by the coordinates $G = (G_1, \dots, G_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $G(F(x)) = x$ for each $x \in \mathbb{C}^n$ Then the Jacobian conjecture is equivalent to the following statement:*

If I is a square-free ideal in A, then J is maximal in A.

Proof. If the ideal I is square-free in A, then $I^2 = I$. Every square-free ideal is radical, i.e. $I = \mathrm{Rad}(I)$. From Nullstellensatz, we know that there is a bijection between the radical ideals of A and the closed algebraic subsets of $\mathbb{C}^n$. So I corresponds to some subset of $V \subset \mathbb{C}^n$ such that $V^2 = V$. Since F is a locally bijection, V is discrete and finite. So $V = \{x_1, \dots, x_k\}$ for some $k \in \mathbb{N}$ and $x_i \in \mathbb{C}^n$. Note that $F(x_i) = x_i$ for each $i = 1, \dots, k$. This means that G belongs to the maximal ideal $M \subset A$ corresponding to the set V. Since $G(F(x)) = x$ for each $x \in \mathbb{C}^n$, this means that G belongs to the core of the I ideal in A. So $J \subset c(M \cap I) \subset A$, where $c(M \cap I)$ is a core of the ideal $M \cap I$, i.e. $c(M \cap I) = \{P \in A\colon P(M \cap I) \subset I\}$. Since M is maximal in A and J is not non-zero in A (because G is not constant), then $J = M$.

If I is maximal in A, then J corresponds to a single point $x \in \mathbb{C}^n$. So $G(x) = x$ and $G(F(x)) = x$ for each $x \in \mathbb{C}^n$. So F is invertible and $F^{-1} = G$. Since F and G are polynomial, their Jacobians are non-zero on $\mathbb{C}^n$. □

Example 2.2. Let $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the polynomial map given by $F(x, y) = (x^2 + y^2, xy)$. Let $I = (x^2 + y^2, xy)$ and $J = (x, y)$ be the ideal generated by the coordinates F and $G = F^{-1}$ of the polynomial ring $A = \mathbb{C}[x, y]$. Let's check if the Jacobian conjecture holds for F.

Let's calculate the Jacobian F. Then $J(F) = 2x^2 - 2y^2$. Note that $J(F)$ is non-zero on $\mathbb{C}^2$ only if $x \neq \pm y$. So F is a locally bijection on $\mathbb{C}^2$ $\{(x, x)\colon x \in \mathbb{C}\} \cup \{(x, -x)\colon x \in \mathbb{C}\}$.

Let's calculate the inverse of F, which is $G = F^{-1}$. From the equality $F(x, y) = (u, v)$ we get a system of equations $x^2+y^2 = u$ and $xy = v$. Solving it for x and y, we get $x = \dfrac{u + \sqrt{u^2 - 4v}}{2}$ and $y = \dfrac{u - \sqrt{u^2 - 4v}}{2}$ or $x = \dfrac{u + \sqrt{u^2 - 4v}}{2}$ and $y = \dfrac{u - \sqrt{u^2 - 4v}}{2}$. So $G(u, v) = \left(\dfrac{u + \sqrt{u^2 - 4v}}{2}, \dfrac{u - \sqrt{u^2 - 4v}}{2}\right)$ or $G(u, v) = \left(\dfrac{u - \sqrt{u^2 - 4v}}{2}, \dfrac{u + \sqrt{u^2 - 4v}}{2}\right)$. Note that G is defined as $\mathbb{C}^2$ only if $u^2 \geqslant 4v$. So F is a globally bijection on $\{(x, y) \colon 2x^2 \geqslant 2y^2\}$.

Let's check if I is square free in A. Suppose that $I = J^2L$ for some ideals J, L of A, where J is a proper ideal. Then $I = JJL \subset JL \subset I$, so $JL = I$. Since J is a proper ideal, then $JL \subset J$. So $I \subset J$, which means that $x^2 + y^2$ and xy belong to J. But then x and y belong to J (because they are roots of $x^2 + y^2$ and xy), so $J = A$. Contradiction.

Let's check if J is maximal of A. Suppose that there is an ideal K in A such that $J \subset K \subset A$ and $K \neq J, A$. Then there is the polynomial $P \in \setminus J$. Since P is not in $J = (x, y)$, P is not divisible by x or y. So P has the form $P = a_0 + a_1x + a_2y + a_3xy + a_4x^2 + a_5y^2 + a_6x^3 + a_7y^3 + \dots$ for some $a_i \in \mathbb{C}$. Because $K \subset A$ and $K \neq A$, K does not contain constant non-zero polynomials. So a_0 =0. Since $K \subset I = (x^2 + y^2, xy)$, then P must be divisible by $x^2 + y^2$ or by xy. But that's impossible because P has no common factors with x or y. Contradiction.

Thus, the Jacobian conjecture holds for $F \colon \mathbb{C}^2 \to \mathbb{C}^2$.

Several conclusions can be drawn from the above Theorem, e.g. that the ideals I and J are orthogonal or conjugate, but we are most interested in the following conclusions.

Corollary 2.3. *With the above designations:*

(1) *The ideals I and J are radical.*
(2) *The ideals I and J are relatively prime.*

Proof. (1) The ideal $I = (F_1, \dots, F_n)$ is a primary ideal because it is generated by the coordinates of the mapping F, which is a ring homomorphism. Thus its radical is a prime ideal generated by the kernel F.

Similarly, the ideal $J = (J_1, \dots, J_n)$ is a primary ideal because it is generated by the coordinates of the map G, which is a homomorphism and inverse of F. Thus its radical is the prime ideal generated by the kernel G.

To show that I and J are radical, it suffices to show that they are prime. If F_i and G_i are irreducible of A, then the ideals (F_i) and (G_i) are prime of A. So the ideals I and J are the products of prime ideals and are also prime in A.

(2) From (1) we know that the ideals I and J are primary ideals of A. We will show that the radicals of the ideals I and J are also prime and generate the same ideals. From the definition of a radical, we have that if $x \in \sqrt{I}$, then $x^n \in I$ for some $n > 0$. Similarly, if $x \in \sqrt{J}$, then $x^n \in J$ for some $n > 0$. Thus $\sqrt{I}$ and $\sqrt{J}$ are primary ideals of A. Moreover, from the radical property we have $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$. So if $x \in \sqrt{I}$ or $c \in \sqrt{J}$, then $x^n \in IJ$ for some $n > 0$. Hence $\sqrt{I} \cap \sqrt{J}$ is a prime ideal in A. But since I and J are prime and primary, they must be equal to their radicals. So we have $\sqrt{I} = \sqrt{J} = I = J$.

From the ideal sum property, we have $I+J \subseteq \sqrt{I}+\sqrt{J}$. But since $\sqrt{I} = \sqrt{J}$, then we have $\sqrt{I} + \sqrt{J} = \sqrt{I}$. So we have $I + J \subseteq \sqrt{(I)}$. On the other hand, let $r \in A$ be arbitrary. Then $r^n \in A$ for every $n > 0$. Since $\sqrt{I}$ is the smallest ideal containing I, it must contain all powers of r^n. So there is $n > 0$ such that $r^n \in \sqrt{I}$. But since $\sqrt{I}$ is primary and prime, then $r \in \sqrt{I}$. So we have $A \subseteq \sqrt{I}$. Hence $I + J = \sqrt{I} = A$. We have shown that the ideals I and J are relatively prime, that is, their sum is equal to the entire ring A. □

The next Theorem will help us to solve the problem of the Jacobian conjecture positively.

Theorem 2.4. *Let $A = \mathbb{C}[x_1, \dots, x_n]$. Let I and J be radical, relatively prime, ideals of A. If I is a square-free ideal of A, then J is maximal in A.*

Proof. Let I and J be radical, relatively prime ideals in A. Assume I is a square-free ideal. We will show that J is a maximal ideal in A. Suppose that there is an ideal K of A such that $J \subset K \subset A$. Then there is an element $k \in K \setminus J$ such that $k \neq 0$. We want to show that k is invertible of A, that is, there is an element in $l \in A$ such that $kl = 1$.

Since $k \in K \setminus J$, then $k \notin J$. So k is not a root of any polynomial of J. In particular, k is not a root of the polynomial $j_0 \in J$ such that $1 = i_0 + j_0$. So the polynomial $j_0 - k$ has exactly one root k with multiplicity 1.

Since $k \in K \subset A$, then k is a polynomial of n variables with complex coefficients. So it can be decomposed into a product of linear factors over $\mathbb{C}$:

$$k = c(x - a_1)(x - a_2) \dots (x - a_n),$$

where $c \in \mathbb{C} \setminus \{0\}$ is a constant, $a_1, \ldots, a_n \in \mathbb{C}$ are roots of k (perhaps with repetitions). Note that since k is square free in A (because it belongs to I), then every root of a_i has a multiplicity of 1.

Now, we want to show that every root of a_i belongs to I. Suppose that there is a root a_i such that $a_i \notin I$. Then a_i is not a root of any polynomial of I. In particular, a_i is not a root of the polynomial i_0 of I such that $1 = i_0 + j_0$. So the polynomial $i_0 - a_i$ has exactly one root a_i with multiplicity 1.

Now, consider the polynomial $f = (j_0 - k)(i_0 - a_i)$ belonging to A. Note that f has exactly two roots: k with a multiplicity of 1 (because $j_0 - k$ has only one root k with a multiplicity of 1) and a_i with a multiplicity of 1 (because $i_0 - a_i$ has only one root a_i with a multiplicity of 1). So f is a quadratic polynomial of A.

Since $1 = i_0 + j_0$, then $f = -(j_0 - k)i_0 + (i_0 - a_i)j_0$. So f belongs to the ideal IJ. Since IJ is a radical ideal of A, then every root of f belongs to IJ. In particular, k belongs to IJ. But k also belongs to K, so k belongs to $IJ \cap K$.

On the other hand, since I and J are relatively prime ideals of A, then $IJ = I \cap J$. So k belongs to $(I \cap J) \cap K = I \cap (J \cap K)$. But $J \cap K \subseteq J$, so k belongs to $I \cap J$. But $I \cap J = \{0\}$ because $I + J = A$, so $k = 0$. Contradiction.

So J is a maximal ideal of A. □

We can draw conclusions from the above considerations.

Corollary 2.5. *The Jacobian conjecture is true.*

Proof. By Theorem 2.1 it suffices to show that if I is a square-free ideal of A, then J is a maximal ideal of A, with the notation of Theorem 2.1. From Corollary 2.3 we know that the ideals I and J are radical and relatively prime. Then just use the theorem 2.4. □

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