The problem of existence and uniqueness of solutions to certain equations describing certain cost and production functions

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Abstract

In this article we present a solution to the problem of existence and uniqueness of solutions to difference and differential equations of cost functions and common production functions.

1 Introduction

Difference and differential equations are useful tools for modeling and analyzing various economic phenomena, such as economic growth, inflation, business cycles, general equilibrium stability, and derivatives pricing. The use of these equations makes it possible to describe the dynamics of economic systems, taking into account uncertainty and shock. However, solving these equations is not a simple or trivial task. It is not always possible to solve such an equation analytically or numerically, or the solution may be ambiguous or non-existent. Therefore, appropriate mathematical methods and conditions are needed to solve difference and differential equations in economics.

In the literature we can find many methods for solving difference and differential equations in economics, such as analytical, numerical or graphical

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methods. However, they are not always effective or available. In addition, the existence and uniqueness of a solution to such an equation or its extendibility cannot always be guaranteed. Therefore, additional conditions and mathematical tools are often used, such as the Lipschitz condition, the Picard theorem, the theorem on the existence of global solutions to differential equations, or the Z-transform. However, these conditions and tools have their limits and scope of applicability. For more information, see [1], where the author introduces mathematical methods for students of economics and finance in a concise and accessible style. This book includes chapters on difference and differential equations and their applications in economics. Cornean in [2] discusses the basics of the theory and methods of solving differential equations and their applications in economics and finance. In the book [3], the authors present mathematical models based on difference and differential equations and show their applications in physics, biology, chemistry and economics. Varian in [6] introduces the basic properties and types of cost functions and alludes to the production and demand functions in relation to the problem of this article. In addition, in the article [4] the author discusses the duality between the cost function and the production function as alternative descriptions of production technologies and their application in economics. We ignore this duality. However, it was taken into account in the article [5], where the problem of securing data of cost and production functions was discussed.

In this article, we present some new theorems concerning the problem of the existence and uniqueness of solutions to difference and differential equations in economics. Our theorems also apply to specific cost and production functions. For simplicity, we will denote the cost and production functions as a simple mathematical function f dependent on certain variables. Thus, the cost function will be denoted by $f(x) = ax + b$, where b is a fixed cost, a is a variable cost depending on the factor x. But nowadays such a function may not be enough, because we can change the prices in certain time intervals $t_0 = 0,\ t_1,\ t_2,\ \ldots,\ t_n$. Therefore, we will also consider the general cost function

$$f(x) = a_0 + \sum_{i=1}^{n} a_i \max(0, \min(t_i - t_{i-1}, x - t_{i-1})),$$

where the general cost function is a linear piecewise function.

In the article we also build on the three most well-known types of production functions. We consider the traditional type of production function:

$$Q(x_1, \ldots, x_n) = a_1x_1 + \ldots a_nx_n.$$

The Cobb-Douglas type production function is often encountered:

$$Q(x_1, \dots, x_n) = a_0 x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}.$$

And also the minimum function:

$$Q(x_1, \dots, x_n) = \min(a_1 x_1, \dots, a_n x_n).$$

Example 1.1. Suppose the production function has the form:

$$Q_t = A K_t^{\alpha} L_t^{\beta},$$

where Q_t is the amount of good produced in the time t, A is a constant technology parameter, K_t is the amount of capital used in the time t, L_t is the amount of labor used in t, and α and β are the parameters of the elasticity of production with respect to capital and labour. Suppose also that capital is subject to the law of accumulation:

$$K_{t+1} = (1 - \delta) K_t + I_t,$$

where δ is the capital amortization factor and I_t is the investment over t. We want to find a difference equation describing the dynamics of production over time. We can do this by substituting the second equation into the first and getting:

$$Q_{t+1} = A \left[(1 - \delta) K_t + I_t\right]^{\alpha} L_{t+1}^{\beta}.$$

This first-order difference equation describes the relationship between production in period $t + 1$ and production and other variables in period t. To solve this equation, we need to assume some form of I_t and L_{t+1}. One possible assumption is that investment and labor are proportional to output, that is:

$$I_t = s Q_t,$$

$$L_{t+1} = n Q_t,$$

where s and n are constant parameters.

Then the difference equation simplifies to:

$$Q_{t+1} = A \left[(1 - \delta) K_t + s Q_t\right]^{\alpha} (n Q_t)^{\beta}.$$

We can try to solve this equation in the form:

$$Q_t = C K_t^{\gamma},$$

where C and γ are constants to be determined. Substituting this solution into the difference equation, we get:

$$CK_{t+1}^{\gamma} = A\left[(1-\delta)K_t + sCK_t^{\gamma}\right]^{\alpha}(nCK_t^{\gamma})^{\beta}.$$

After simplifying and dividing by K_{t+1}^{γ}, we get:

$$C = A\left[(1-\delta) + sC\right]^{\alpha}(nC)^{\beta}K_{t+1}^{-\gamma}.$$

For this equation to hold for any K_{t+1}, the following condition must be met:

$$\gamma = \alpha + \beta$$

and

$$C = A\left[(1-\delta) + sC\right]^{\alpha}(nC)^{\beta}.$$

The latter equation is a non-linear equation with respect to C that can have zero, one, or more solutions. If there are no solutions, then there is no solution to the difference equation. If there is more than one solution, then there is more than one solution to the difference equation. Only when there is exactly one solution to C is there exactly one solution to the difference equation.

This example shows that the problem of the existence and uniqueness of solutions to difference equations in economics can be difficult and depends on the assumptions about the investment and labor functions and on the values of technology and behavior parameters.

Here is a similar example, but for a differential equation:

Example 1.2. Suppose the production function has the form:

$$Q_t = AK_t^{\alpha}L_t^{\beta},$$

where Q_t is the amount of good produced in the time t, A is a constant technology parameter, K_t is the amount of capital used in t, L_t is the amount of labor used in t, and α and β are the parameters of the elasticity of production with respect to capital and labour. Suppose also that capital is subject to the law of accumulation:

$$\frac{dK_t}{dt} = I_t - \delta K_t,$$

where δ is the capital amortization factor and I_t is the investment over t. We want to find a differential equation describing the dynamics of production over time. We can do this by substituting the first equation into the second and getting:

$$\frac{dQ_t}{dt} = A\left(\alpha K_t^{\alpha-1} L_t^{\beta} \frac{dK_t}{dt} + \beta K_t^{\alpha} L_t^{\beta-1} \frac{dL_t}{dt}\right).$$

This first-order differential equation describes the relationship between a change in output and a change in capital and labour. To solve this equation, we need to assume some form of I_t and $\frac{dL_t}{dt}$. One possible assumption is that investment and labor are proportional to output, that is:

$$I_t = sQ_t,$$
$$\frac{dL_t}{dt} = nQ_t,$$

where s and n are constant parameters. Then the differential equation simplifies to:

$$\frac{dQ_t}{dt} = A\left(\alpha s + \beta n\right) Q_t^{\alpha+\beta}.$$

We can try to solve this equation in the form:

$$Q_t = Ce^{rt},$$

where C and r are constants to be determined. Substituting this solution into the differential equation, we get:

$$Cre^{rt} = A\left(\alpha s + \beta n\right) C^{\alpha+\beta} e^{(\alpha+\beta)rt}.$$

After simplifying and dividing by Ce^{rt}, we get:

$$r = A\left(\alpha s + \beta n\right) C^{\alpha+\beta-1}.$$

For this equation to hold for any t, the following condition must be met:

$$\alpha + \beta - 1 = 0$$

and

$$r = A\left(\alpha s + \beta n\right) C^{0} = A\left(\alpha s + \beta n\right).$$

The latter equation is a linear equation with respect to C that has exactly one solution. Then there is exactly one solution to the differential equation. We can try to solve C algebraically by solving the linear equation:

$$C = \frac{r}{A(\alpha s + \beta n)}.$$

This example shows that the problem of the existence and uniqueness of solutions to differential equations in economics can be easier or more difficult depending on the assumptions about the investment and labor functions and on the values of technology and behavior parameters.

The aim of this paper is to solve the problem of the existence and uniqueness of solutions to the difference and differential equations describing the cost and production functions that are most common in many enterprises.

In sections 2 and 3 we focus on the difference equations describing the cost and production functions. In sections 4 and 5 we discuss differential equations. Particularly noteworthy are Theorems 5.5 and 5.6, where the minimum production function negates the existence of a solution.

2 Difference equations describing cost functions

In this section we will focus on recursive cost functions in the traditional form, i.e. those whose solutions are linear functions and linear piece functions over certain time intervals.

Theorem 2.1. *Let $f(n) = a + f(n-1)$, where $f(0) = b$ for some constants a and b, be an equation describing the cost function. Then f has a unique solution.*

Proof. We will show by mathematical induction that $f(n) = an + b$ is a unique solution to the difference equation $f(n) = a + f(n-1)$.

For $n = 0$ we have $f(0) = b$.
Suppose that for every $k \geqslant 0$, $f(k) = ak + b$ holds. We will show that it also holds for $k + 1$. We have:

$$f(k+1) = a + f(k) = a + ak + b = a(k+1) + b.$$

The solution is obviously unambiguous. □

Now let's move on to a function more real in the modern world, but we'll start with the one-time case t_1.

Theorem 2.2. *Let's*

$$f(n+1) = \begin{cases} f(n) + a_1 & \textit{for} \quad 0 \leqslant n < t_1 \\ f(n) + a_2 & \textit{for} \quad t_1 \leqslant n \end{cases}$$

be an equation describing the cost function, where $f(0) = a_0$*,* $f(t_1) = a_0 + a_1 t_1$ *for some constants* a_0*,* a_1*,* a_2*,* t_1*. Then* f *has a solution that is unambiguous.*

Proof. We will show by mathematical induction that

$$f(n) = \begin{cases} a_0 + a_1 n & \text{for} \quad 0 \leqslant n < t_1 \\ a_0 + a_1 t_1 + a_2 n & \text{for} \quad t_1 \leqslant n \end{cases}$$

is a unique solution to the difference equation

$$f(n+1) = \begin{cases} f(n) + a_1 & \text{for} \quad 0 \leqslant n < t_1 \\ f(n) + a_2 & \text{for} \quad t_1 \leqslant n \end{cases}$$

For $n = 0$ we have $f(0) = a_0$ and for $n = t_1$ we have $f(t_1) = a_0 + a_1 t_1$. Assume that for every $k \geqslant 0$ holds

$$f(n) = \begin{cases} a_0 + a_1 n & \text{for} \quad 0 \leqslant n < t_1 \\ a_0 + a_1 t_1 + a_2 n & \text{for} \quad t_1 \leqslant n \end{cases}$$

We will show that it also holds for $k + 1$. For $0 \leqslant k < k + 1 < t_1$ we have

$$f(k+1) = f(k) + a_1 = a_0 + a_1 k + a_1 = a_0 + a_1(k+1)$$

For $t_1 \leqslant k < k + 1$ we have

$$f(k+1) = f(k) + a_2 = a_0 + a_1 t_1 + a_2 k + a_2 = a_0 + a_1 t_1 + a_2(k+1)$$

By the principle of mathematical induction, we have proved our solution, which is unambiguous.

Moreover, we note that our solution can be written as:

$$f(n) = a_0 + a_1 \min(t_1, n) + a_2 \max(0, n - t_1),$$

which is a linear piece cost function. □

The next function is more difficult, but the most realistic at the moment.

Theorem 2.3. *Let*

$$f(m+1) = \begin{cases} f(m)+a_1 & \textit{for} \quad 0 \leqslant m < t_1 \\ f(m)+a_2 & \textit{for} \quad t_1 \leqslant m < t_2 \\ \dots & \dots \quad \dots \\ f(m)+a_{n-1} & \textit{for} \quad t_{n-2} \leqslant m < t_{n-1} \\ f(m)+a_n & \textit{for} \quad t_{n-1} \leqslant m \end{cases}$$

be a difference equation describing the cost function, where $f(0) = a_0$*,* $f(t_i) = f(t_{i-1}) + (t_i - t_{i-1})a_i$*, where* $i = 1, 2, \dots, m$ *and for some constants* a_0*,* a_1*,* $\dots$*,* a_n*,* t_1*,* t_2*,* $\dots$*,* t_n*. Then there is a solution and it is unambiguous.*

Proof. We will show by mathematical induction that

$$f(m) = \begin{cases} a_0 + a_1 n & \text{for} \quad 0 \leqslant m < t_1 \\ a_0 + \sum_{j=1}^{i-1} \max(0, \min(t_j - t_{j-1}, x - t_{j-1})) + & \text{for} \quad t_{i-1} \leqslant m < t_i \\ \quad +a_i \max(0, x - t_{i-1}) & \\ a_0 + \sum_{j=1}^{n} a_j (t_j - t_{j-1}) + a_n (x - t_n) & \text{for} \quad t_n < m \end{cases}$$

is a unique solution to the difference equation

$$f(m+1) = \begin{cases} f(m)+a_1 & \text{for} \quad 0 \leqslant m < t_1 \\ f(m)+a_2 & \text{for} \quad t_1 \leqslant m < t_2 \\ \dots & \dots \quad \dots \\ f(m)+a_{n-1} & \text{for} \quad t_{n-2} \leqslant m < t_{n-1} \\ f(m)+a_n & \text{for} \quad t_{n-1} \leqslant m \end{cases}$$

For $n = 0$ we have $f(0) = a_0$ and for $n = t_i$ we have

$$f(t_i) = a_0 + \sum_{j=1}^{i-1} \max(0, \min(t_j - t_{j-1}, t_i - t_{j-1})) + a_i \max(0, t_i - t_{i-1}) =$$
$$= a_0 + \sum_{j=1}^{i-1} \max(0, t_j - t_{j-1}) + a_i (t_i - t_{i-1}) = f(t_{i-1}) + a_i (t_i - t_{i-1}).$$

Assume that for every $k \geqslant 0$ holds

$$f(k)=\begin{cases} a_0+a_1k & \text{for} \quad 0\leqslant k<t_1 \\ a_0+\sum\limits_{j=1}^{i-1}\max(0,\min(t_j-t_{j-1},k-t_{j-1}))+ & \text{for} \quad t_{i-1}\leqslant k<t_i \\ \quad +a_i\max(0,k-t_{i-1}) & \\ a_0+\sum\limits_{j=1}^{n}a_j(t_j-t_{j-1})+a_n(k-t_n) & \text{for} \quad t_n<m \end{cases}$$

We will show that it also holds for $k+1$. For $0\leqslant k<k+1<t_1$ we have

$$f(k+1)=f(k)+a_1=a_0+a_1k+a_1=a_0+a_1(k+1).$$

For $t_1\leqslant k<k+1<t_2$ we have

$$f(k+1)=f(k)+a_2=a_0+a_1t_1+a_2k+a_2=a_0+a_1t_1+a_2(k+1).$$

We prove similarly for the next intervals.

By the principle of mathematical induction, we have proved our solution, which is unambiguous.

Moreover, we note that our solution can be written as:

$$f(n)=a_0+\sum_{i=1}^{n}a_i\max(0,\min(t_i-t_{i-1},x-t_{i-1})),$$

which is a linear piece cost function. □

3 Difference equations describing production functions

In this section, we will present the difference equations describing the production functions. Due to the ambiguity of the choice of production functions by many enterprises, the most common types of functions are included in this section.

We will start with the most popular function.

Theorem 3.1. *Let*

$$\begin{cases} f(x_1+1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + a_1 \\ f(x_1, x_2+1, x_3, \dots, x_n) = f(x_1, x_2, \dots, x_n) + a_2 \\ \dots\dots\dots \\ f(x_1, \dots, x_{n-1}, x_n+1) = f(x_1, \dots, x_n) + a_n \end{cases}$$

be a system of difference equations describing the production function, where $f(0, \dots, 0) = 0$*, where* a_1*,* a_2*,* $\dots$*,* a_n *are some constants. Then there is a solution and it is unambiguous.*

Proof. We will show by mathematical induction that the function $f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n$ is a solution to the system of difference equations:

$$\begin{cases} f(x_1+1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + a_1 \\ f(x_1, x_2+1, x_3, \dots, x_n) = f(x_1, x_2, \dots, x_n) + a_2 \\ \dots\dots\dots \\ f(x_1, \dots, x_{n-1}, x_n+1) = f(x_1, \dots, x_n) + a_n \end{cases}$$

Of course, $f(0, \dots, 0) = 0$. Assume our solution is true for the system $(x_1, \dots, x_n)$. We will show that the solution is true for the system $(x_1 + 1, x_2, \dots, x_n)$. We have:

$$f(x_1+1, x_2, \dots, x_n) = f(x_1, \dots, x_n) + a_1 = a_1x_1 + \dots + a_nx_n + a_1 =$$
$$= a_1(x_1+1) + a_2x_2 + \dots + a_nx_n$$

We prove similarly for systems $(x_1, x_2+1, x_3, \dots, x_n)$, $\dots$, $(x_1, \dots, x_{n-1}, x_n+1)$.

By the principle of mathematical induction, the function $f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n$ is a solution to the difference system and is unambiguous. □

The next function, or rather a set of functions, is a derivative of the Cobb-Douglas type production function, which allows us to examine various factors of production x_1, x_2, $\dots$, x_n.

Theorem 3.2. *Let*

$$\begin{cases} f(x_1+1, x_2, \dots, x_n) = f(x_1, \dots, x_n) + \left(\dfrac{x_1+1}{x_1}\right)^{a_1} \\ f(x_1, x_2+1, x_3, \dots, x_n) = f(x_1, \dots, x_n) + \left(\dfrac{x_2+1}{x_2}\right)^{a_2} \\ \dots\dots\dots \\ f(x_1, \dots, x_{n-1}, x_n+1) = f(x_1, \dots, x_n) + \left(\dfrac{x_n+1}{x_n}\right)^{a_n}. \end{cases}$$

be a set of recursive equations describing the production function where $f(0,\ldots,0) = 0$*, for some constants* a_1*,* a_2*, …,* a_n*. Then there is a solution and it is unambiguous.*

Proof. We will show by mathematical induction that the function $f(x_1,\ldots,x_n) = a_0 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is a solution to the system of recursive equations:

$$\begin{cases} f(x_1+1, x_2, \ldots, x_n) = f(x_1,\ldots,x_n) + \left(\dfrac{x_1+1}{x_1}\right)^{a_1} \\ f(x_1, x_2+1, x_3, \ldots, x_n) = f(x_1,\ldots,x_n) + \left(\dfrac{x_2+1}{x_2}\right)^{a_2} \\ \qquad \cdots\cdots\cdots \\ f(x_1, \ldots, x_{n-1}, x_n+1) = f(x_1,\ldots,x_n) + \left(\dfrac{x_n+1}{x_n}\right)^{a_n}. \end{cases}$$

Of course, $f(0,\ldots,0) = 0$. Assume our solution is true for the system $(x_1,\ldots,x_n)$. We will show that the solution is true for the system $(x_1 + 1, x_2, \ldots, x_n)$. We have:

$$f(x_1+1, x_2, \ldots, x_n) = f(x_1,\ldots,x_n) + \left(\frac{x_1+1}{x_1}\right)^{a_1} =$$
$$= a_0 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} + \left(\frac{x_1+1}{x_1}\right)^{a_1} = a_0 (x_1+1)^{a_1} x_2^{a_2} \ldots x_n^{a_n}.$$

We prove similarly for systems $(x_1, x_2+1, x_3, \ldots, x_n)$, …, $(x_1, \ldots, x_{n-1}, x_n + 1)$.

By the principle of mathematical induction, the function $f(x_1,\ldots,x_n) = a_0 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is a solution to the difference system and is unambiguous. □

Finally, one more function for companies where they are not interested in individual production factors, but which production factor is the weakest.

Theorem 3.3. *Let*

$$f(x_1,\ldots,x_i{+}1,\ldots,x_n) = \begin{cases} f(x_1,\ldots,x_i,\ldots,x_n) + a_i & \text{for } \; a_i x_i < a_j x_j, j \neq i \\ f(x_1,\ldots,x_i,\ldots,x_n) & \text{for } \; a_i x_i > a_j x_j, j \neq i \end{cases}$$

for $i = 1,\ldots,n$*, where* $f(0,\ldots,0) = 0$ *and for some constants* a_1*,* a_2*, …,* a_n*, be a system of difference equations describing the production function. Then there is a solution and it is unambiguous.*

Proof. We will show by mathematical induction that the function $f(x_1, \ldots, x_n) = \min(a_1x_1, \ldots, a_nx_n)$ is a solution to the system of recursive equations:

$$f(x_1, \ldots, x_i{+}1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_i, \ldots, x_n) + a_i & \text{for} \quad a_ix_i < a_jx_j, j \neq i \\ f(x_1, \ldots, x_i, \ldots, x_n) & \text{for} \quad a_ix_i > a_jx_j, j \neq i \end{cases}$$

for $i = 1, \ldots, n$.

Of course, $f(0, \ldots, 0) = 0$. Assume our solution is true for the system $(x_1, \ldots, x_n)$. We will show that the solution is true for the system $(x_1 + 1, x_2, \ldots, x_n)$. We have two cases:

$$f(x_1 + 1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + a_1 = \min(a_1x_1, \ldots, a_nx_n) + a_1 =$$
$$= \min(a_1(x_1 + 1), a_2x_2, \ldots, a_nx_n),$$

for $a_1x_1 < a_ix_i$, where $i \neq 1$, and

$$f(x_1 + 1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) = \min(a_1x_1, \ldots, a_nx_n),$$

for $a_1x_1 > a_ix_i$, where $i \neq 1$.

We prove similarly for systems $(x_1, x_2 + 1, x_3, \ldots, x_n)$, ..., $(x_1, \ldots, x_{n-1}, x_n + 1)$.

By the principle of mathematical induction, the function $f(x_1, \ldots, x_n) = \min(a_1x_1, \ldots, a_nx_n)$ is a solution to the difference system and is unambiguous. □

4 Differential equations describing cost functions

In mathematics, the derivative is often a function that describes the motion of some process (economic, physical, etc.). In this section, as in the section 2, we will focus on linear type cost functions and linear slices.

Theorem 4.1. *Let*

$$f'(x) = a,$$

with $f(0) = b$, where a and b are some constants, be a differential equation describing the cost function. Then there is a solution to such an equation that is unique.

Proof. Let's solve the differential equation. After integrating, we get:

$$f(x) = ax + C,$$

where C is some constant. So the general solution is a linear function with parameters a and C.

The solution is clear. The initial condition is $f(0) = b$. So $C = b$. Then the special solution is of the form:

$$f(x) = ax + b.$$

□

And now we will present a piecewise linear function for one change in time t_1.

Theorem 4.2. *Let*

$$f'(x) = \begin{cases} a_1 & \textit{for} \quad 0 \leqslant x < t_1 \\ a_2 & \textit{for} \quad x \geqslant t_1 \end{cases}$$

be a differential equation describing the cost function, where $f(0) = a_0$, $f(t_1) = a_0 + a_1 t_1$, for some constants a_0, a_1, a_2, t_1. Then there is a solution to this equation that is unique.

Proof. There is a general solution, after integrating f' for each interval we get:

$$f(x) = \begin{cases} a_1 x + C_1 & \text{for} \quad 0 \leqslant x < t_1 \\ a_2 x + C_2 & \text{for} \quad x \geqslant t_1 \end{cases}$$

where C_1, C_2 are constants. Thus the general solution is in the form of a linear piece function.

Our initial conditions are $f(0) = a_0$ and $f(t_1) = a_0 + a_1 t_1$. After substituting, we get $C_1 = a_0$ and $C_2 = a_0 - (a_2 - a_1)t_1$. Then the special solution is of the form:

$$f(x) = \begin{cases} a_0 + a_1 x & \text{for} \quad 0 \leqslant x < t_1 \\ a_0 - (a_2 - a_1)t_1 + a_2 x & \text{for} \quad x \geqslant t_1 \end{cases}$$

We can write this solution as

$$f(x) = a_0 + a_1 \min(t_1, x) + a_2 \max(0, x - t_1).$$

□

At the end of this section, we present the most viable differential function describing the cost function for n changes over time.

Theorem 4.3. *Let*

$$f'(x) = \begin{cases} a_1 & \textit{for} \quad 0 \leqslant x < t_1 \\ a_2 & \textit{for} \quad t_1 \leqslant x < t_2 \\ \dots & \dots \qquad \dots \\ a_n & \textit{for} \quad x \geqslant t_{n-1} \end{cases}$$

be a differential equation describing the cost function, with $f(0) = a_0$, $f(t_i) = f(t_{i-1}) + (t_i - t_{i-1})a_i$, *where* $i = 1, 2, \dots, m$, *for some* a_0, a_1, …, a_n, t_0, t_1, …, t_{n-1}. *Then there is a solution that is unambiguous.*

Proof. There is a general solution, after integrating f' for each interval we get:

$$f(x) = \begin{cases} a_1 x + C_1 & \text{for} \quad 0 \leqslant x < t_1 \\ a_2 x + C_2 & \text{for} \quad t_1 \leqslant x < t_2 \\ \dots & \dots \qquad \dots \\ a_n x + C_n & \text{for} \quad t_{n-1} \leqslant x \end{cases}$$

where C_1, C_2, …, C_n are constants. Thus the general solution is in the form of a linear piece function.

Our initial conditions are $f(0) = a_0$ and $f(t_i) = f(t_{i-1}) + (t_i - t_{i-1})a_i$, where $i = 1, 2, \dots, m$.

To find a particular solution to the differential equation for a piecewise linear function on n pieces, solve a system of $2n$ equations with $2n$ unknowns C_i and a_i. Such a solution is always unique by virtue of the theorem on the existence and uniqueness of the solution to the Cauchy problem for ordinary differential equations.

After calculations, we should get a special solution of the form:

$$f(x) = a_0 + \sum_{i=1}^{n} a_i \max(0, \min(t_i - t_{i-1}, x - t_{i-1})).$$

□

5 Differential equations describing production functions

In this section, we will present the differential equations describing the production functions. Due to the ambiguity of the production function selection, the most common types of functions will also be included.

Theorem 5.1. *Let*

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = a_1 \\ \dfrac{\partial f}{\partial x_2} = a_2 \end{cases}$$

be a system of partial differential equations describing the production function, where $f(0,0) = 0$ and a_1, a_2 are some constants. Then there is a solution that is unambiguous.

Proof. To solve this system, we need to find a f function that satisfies both conditions. Integrating the first equation with respect to x_1 we get:

$$f(x_1, x_2) = a_1 x_1 + C(x_2),$$

where $C(x_2)$ is any function that depends on x_2. Integrating the second equations with respect to x_2 we get:

$$f(x_1, x_2) = a_2 x_2 + D(x_1),$$

where $D(x_1)$ is any function that depends on x_1.

Now we need to reconcile both solutions, so we have:

$$a_1 x_1 + C(X_2) = a_2 x_2 + D(x_1).$$

For this equation to be true for any values of x_1, x_2 must have $a_1 = D'(x_1)$ and $a_2 = C'(x_2)$.

Since a_1 and a_2 are constants, then D' and C' are also constants. Therefore, the functions D and C must be linear, in the form:

$$D(x_1) = b_1 x_1 + b_0,$$

$$C(x_2) = b_2 x_2 + b_0',$$

where b_i are any constants.

Substituting these functions into our equation, we get:

$$a_1 x_1 + b_2 x_2 + b_0' = a_2 x_2 + b_1 x_1 + b_0.$$

Arranging like terms, we get:

$$(a_1 - b_1) x_1 + (b_2 - a_2) x_2 + (b_0' - b_0) = 0.$$

For this equation to be true for any values of x_i we must have:

$$a_1 - b_1 = 0 \Rightarrow a_1 = b_1 b_2 - a_2 = 0 \Rightarrow b_2 = a_2 b_0' - b_0 = 0 \Rightarrow b_0' = b_0.$$

Putting these values into our D and C functions, we get:

$$D(x_1) = a_1x_1 + b_0C(x_2) = a_2x_2 + b_0.$$

Putting these functions into our solution $f(x_1, x_2)$ we get (where $b_0 = E$):

$$f(x_1, x_2) = a_1x_1 + a_2x_2 + E.$$

This is the final solution to our differential equation.

Our initial condition is $f(0, 0) = 0$, so $E = 0$. Then the unique special solution is the function:

$$f(x_1, x_2) = a_1x_1 + a_2x_2.$$

□

Now we will present a more generalized version.

Theorem 5.2. *Let*

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = a_1 \\ \dfrac{\partial f}{\partial x_2} = a_2 \\ \dots\dots\dots \\ \dfrac{\partial f}{\partial x_n} = a_n \end{cases}$$

be a system of partial differential equations describing the production function, where $f(0, \dots, 0) = 0$ and a_1, a_2, $\dots$, a_n are some constants. Then there is a solution to the differential equation that is unique.

Proof. To solve this system of differential equations, we need to find a function f that satisfies the above conditions. Integrating the i-th equation with respect to x_i, for $i = 1, 2, \dots, n$, we get:

$$f(x_1, x_2, \dots, x_n) = a_ix_i + C(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

where $C(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ is any function dependent on x_1, $\dots$, x_{i-1}, x_{i+1}, $\dots$, x_n.

We proceed in the same way as in the previous theorem to arrive at the general solution:

$$f(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n + E,$$

where E is any constant. This is the final solution to our differential equation.

Our initial condition is $f(0, 0, \ldots, 0) = 0$, so $E = 0$. Then the unique special solution is the function:

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

□

Now we will introduce a Cobb-Douglass function.

Theorem 5.3. *Let*

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = a_0 a_1 x_1^{a_1 - 1} x_2^{a_2} \\ \dfrac{\partial f}{\partial x_2} = a_0 a_2 x_1^{a_1} x_2^{a_2 - 1} \end{cases}$$

be a system of differential equations describing the production function, where $f(0, 0) = 0$ *and* a_0*,* a_1*,* a_2 *are some constants. Then there is a solution that is unambiguous.*

Proof. To find a solution to this system of equations, we need to integrate both equations with respect to the respective variables. Integrating the first equation with respect to x_1 we get:

$$f(x_1, x_2) = a_0 x_1^{a_1} x_2^{a_2} + C(x_2),$$

where $C(x_2)$ is a constant that depends on x_2. Integrating the second equation with respect to x_2 we get:

$$f(x_1, x_2) = a_0 x_1^{a_1} x_2^{a_2} + D(x_1),$$

where $D(x_1)$ is a constant that depends on x_1. Comparing both sides of the equation, we get:

$$C(x_2) = D(x_1) = E,$$

where E is an integration constant independent of x_1 and x_2. Thus, the solution to this system of partial differential equations is the function $f(x_1, x_2) = a_0 x_1^{a_1} x_2^{a_2} + E$.

Of course, $f(0, 0) = 0$, so the antiderivative is a unique solution

$$f(x_1, x_2) = a_0 x_1^{a_1} x_2^{a_2}.$$

□

Theorem 5.4. *For $i = 1, 2, \dots, n$ let*

$$\frac{\partial f}{\partial x_i} = a_0 a_i x_1^{a_1} \dots x_i^{a_i - 1} \dots x_n^{a_n}$$

be a system of partial differential equations describing the production function, where $f(0, \dots, 0) = 0$ and a_0, a_1, ..., a_n be some constants. Then there is a solution that is unambiguous.

Proof. Proceeding in the same way as in the theorem 5.3 we obtain a general solution and an unambiguous special solution, namely:

$$f(x_1, x_2, \dots, x_n) = a_0 x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} + E,$$
$$f(x_1, x_2, \dots, x_n) = a_0 x_1^{a_1} x_2^{a_2} \dots x_n^{a_n},$$

where E is a constant. □

The following system of differential equations describes the production function, where the company focuses only on the factor of production that performs the worst. In the next theorem it is in the generalized version.

Theorem 5.5. *Let*

$$\begin{cases} \frac{\partial f}{\partial x_1} = a_1 & \text{for} \quad a_1 x_1 < a_2 x_2 \\ \frac{\partial f}{\partial x_1} = 0 & \text{for} \quad a_1 x_1 > a_2 x_2 \\ \frac{\partial f}{\partial x_2} = 0 & \text{for} \quad a_1 x_1 < a_2 x_2 \\ \frac{\partial f}{\partial x_2} = a_2 & \text{for} \quad a_1 x_1 > a_2 x_2 \end{cases}$$

be a system of differential equations describing the production function, where $f(0, \dots, 0) = 0$ and a_1, a_2 are some constants. Then for $a_1 x_1 = a_2 x_2$ the differential equation does not exist. There is also no general solution for the above system of differential equations. In addition, there is a unique solution specific to the points (x_1, x_2) satisfying the condition $a_1 x_1 \neq a_2 x_2$.

Proof. We will show that for points (x_1, x_2) satisfying the condition $a_1 x_1 \neq a_2 x_2$ the function $f(x_1, x_2) = \min(a_1 x_1, a_2 x_2)$ is a special solution.

Note that $f(x_1, x_2) = \min(a_1 x_1, a_2 x_2)$ is not differentiable at (x_1, x_2) such that $a_1 x_1 = a_2 x_2$. In the remaining points, the differential equation describing the function $f(x_1, x_2)$ looks like:

$$
\begin{cases}
\dfrac{\partial f}{\partial x_1} = a_1 \quad \text{for} \quad a_1x_1 < a_2x_2 \\
\dfrac{\partial f}{\partial x_1} = 0 \quad \text{for} \quad a_1x_1 > a_2x_2 \\
\dfrac{\partial f}{\partial x_2} = 0 \quad \text{for} \quad a_1x_1 < a_2x_2 \\
\dfrac{\partial f}{\partial x_2} = a_2 \quad \text{for} \quad a_1x_1 > a_2x_2
\end{cases}
$$

Since the function $f(x_1, x_2)$ is not differentiable at every point, there is no general solution for this system of partial differential equations.

Moreover, if $a_1x_2 < a_2x_2$, then the general solution be a function

$$f(x_1, x_2) = a_1x_1 + C_1,$$

where C_1 is a constant. However, if $a_1x_2 > a_2x_2$, then the general solution be a function

$$f(x_1, x_2) = a_2x_2 + C_2,$$

where C_2 is a constant.

From the initial condition $f(0, 0) = 0$ it follows that the special solutions should be the same functions, with $C_1 = C_2 = 0$. Thus, if $a_1x_1 < a_2x_2$, then

$$f(x_1, x_2) = a_1x_1$$

and if $a_2x_2 > a_1x_1$, then

$$f(x_1, x_2) = a_2x_2.$$

So the above solution can be written as

$$f(x_1, x_2) = \min(a_1x_1, a_2x_2).$$

□

Theorem 5.6. *For $i, j = 1, 2, \ldots, n$, $i \neq j$, let*

$$\frac{\partial f}{\partial x_i} = a_i \quad \textit{for} \quad a_ix_i < a_jx_j,$$

be a system of differential equations describing the production function. For $a_1x_1 = a_2x_2 = \cdots = a_nx_n$ there is no differential equation. There is also no general solution for the above system of differential equations. Moreover, there is a unique solution specific to points $(x_1, x_2, \ldots, x_n)$, where the function reaches a minimum only at one point a_ix_i.

Proof. We solve similarly as in the previous theorem. □

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